



ELSEVIER

Journal of Geometry and Physics 43 (2002) 241–292

JOURNAL OF
GEOMETRY AND
PHYSICS

www.elsevier.com/locate/jgp

Superconnections, anomalies and non-BPS brane charges

Richard J. Szabo^{a,b,*}

^a Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK

^b The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Received 9 October 2001

Abstract

The properties of brane–antibrane systems and systems of unstable D-branes in Type II superstring theory are investigated using the formalism of superconnections. The low-energy open string dynamics is shown to be probed by generalized Dirac operators. The corresponding index theorems are used to compute the chiral gauge anomalies in these systems, and hence their gravitational and Ramond–Ramond couplings. A spectral action for the generalized Dirac operators is also computed and shown to exhibit precisely the expected processes of tachyon condensation on the brane worldvolumes. The Chern–Simons couplings are thereby shown to be naturally related to Fredholm modules and bivariant K-theory, confirming the expectations that D-brane charge is properly classified by K-homology.

© 2002 Elsevier Science B.V. All rights reserved.

MSC: 55B20; 58B50

Subj. Class.: Differential geometry; General relativity

Keywords: Space splitting; Null hypersurfaces; Extrinsic curvature

1. Introduction and summary

The surge of interest in recent years in the study of unstable systems of D-branes (see [1] for reviews) has been sparked in part because they carry information about the non-perturbative vacuum of string theory. The instability present in such systems is marked by the appearance of tachyonic modes in the open string sectors. It was originally suggested that the tachyon field on these D-branes should be properly understood as a Higgs field [2],

* Present address: Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK. Tel.: +44-131-451-3230; fax: +44-131-451-3249.

E-mail address: r.j.szabo@ma.hw.ac.uk (R.J. Szabo).

and that the instability is due to a perturbative expansion about an unstable extremum of the tachyon potential. Assuming certain properties of the potential, the system may then decay into its true vacuum state and sometimes leave behind a topological defect which is indistinguishable from a stable, supersymmetric D-brane. The assumptions which go into this argument have been subsequently verified within the framework of string field theory [3–6] and of worldsheet σ -model effective actions [7].

The prototypical system consists of a collection of branes and antibrane. It has been used to develop the topological classification of D-brane charges in terms of the cohomology of their Chan–Paton gauge bundles, namely K-theory [8,9] (see [10] for a review). In this setting, Ramond–Ramond (RR) charge is characterized by the formal difference of the vector bundles carried by the branes and antibrane. There is a canonical map from the K-theory group into cohomology given by the Chern character, which can be used to give explicit charge formulas for the coupling of Ramond–Ramond potentials to the D-brane worldvolumes. For this, the gauge connections should be replaced by the appropriate analogues for virtual bundles. It was suggested in [9] that the appropriate geometric extension is given by a superconnection [11–13] which naturally incorporates the tachyon field of the unstable system. This description has been exploited recently to describe many aspects of the couplings of unstable D-branes to closed string supergravity fields [5,6,14–17].

In this paper we will present a detailed, mathematical exposition of these relationships. We will pay particular attention to the systematic derivation of the couplings of the unstable D-branes to the Ramond–Ramond tensor potentials. Our analysis will rely on the identifications of anomalies in the brane worldvolume quantum field theories due to the presence of chiral fermion fields. The RR charges of these systems may then be determined by the appropriate modification of standard index theoretical techniques. Such interpretations have also been pointed out in [5,16].

Recall that the RR couplings on the worldvolume Σ of N coincident D-branes in Type II superstring theory is expressed through the Wess–Zumino action [8,18–21]

$$S_{WZ} = \int_{\Sigma} C \wedge \text{tr}_N e^{(F_A - B)/2\pi i} \wedge e^{d/2} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}, \tag{1.1}$$

where throughout we shall work in appropriate string units. In (1.1), C is the pullback of the total RR form potential C under the worldvolume embedding $\phi : \Sigma \rightarrow X$ into the space–time manifold X , F_A the field strength of the $U(N)$ gauge field A on the branes, with tr_N the trace in the fundamental representation of $U(N)$ and B the pullback of the NS–NS two-form potential B . The quantities R_T and R_N are the curvatures of the tangent and normal bundles to Σ in X , respectively, $\hat{A}(R)$ is the usual Dirac index and d is a degree 2 characteristic class which defines a spin^c -structure on Σ . This requires the NS–NS B -field to be topologically trivial [9,22,23]. Action (1.1) can be written in terms of space–time quantities alone in the form [8]

$$S_{WZ} = \int_X C \wedge \text{ch}(\phi_! E) \wedge \sqrt{\hat{A}(TX)}, \tag{1.2}$$

where $\phi_!$ is the induced Gysin map acting on K-theory, E the Chan–Paton gauge bundle supported by the D-branes and ch denotes the usual Chern character. This leads to an

interpretation of RR charges as elements $x = \phi_! E \in K^0(X)$ of the K-theory of space–time. A similar property is true of the Ramond–Ramond fields themselves [24,25]. For non-abelian gauge bundles, the couplings (1.1) and (1.2) require appropriate modifications in order that the action be T-duality-invariant [26,27].

For a system of N^+ coincident branes and N^- coincident antibranes wrapping a submanifold Σ of space–time, we will derive the charge formula

$$\begin{aligned}
 S_{\text{WZ}}^{\text{D}\bar{\text{D}}} = \int_{\Sigma} C \wedge \text{tr}_{N^+ \oplus N^-} \begin{pmatrix} \mathbb{1}_{N^+} & 0 \\ 0 & -\mathbb{1}_{N^-} \end{pmatrix} \exp \frac{1}{2\pi i} \begin{pmatrix} F_{A^+} + T^\dagger T & (DT)^\dagger \\ DT & F_{A^-} + TT^\dagger \end{pmatrix} \\
 \wedge e^{-B/2\pi i} \wedge e^{d/2} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}, \tag{1.3}
 \end{aligned}$$

where F_{A^\pm} are the field strengths of the $U(N^\pm)$ gauge fields A^\pm on the branes and antibranes, respectively, T is the bi-fundamental $\mathbf{N}^- \otimes \mathbf{N}^+$ tachyon field and the gauge-covariant derivative is given by $DT = dT + A^-T + TA^+$. The gauge field part of this action in the case $N^+ = N^-$ was obtained to leading orders in powers of the tachyon field in [14] by a tree-level calculation of the open string effective action on the brane–antibrane system. As proposed there, the full gauge coupling can be elegantly expressed in terms of a superconnection associated with the open string fields A^\pm and T , and indeed within the present formalism this is how these terms shall emerge. For the appropriate Higgs profile of the tachyon field, these couplings were shown to yield the anticipated RR couplings to lower-dimensional D-branes that remain after tachyon condensation. We will show that the same is true when one includes the gravitational couplings to the brane worldvolume, as in (1.3). Action (1.3) also has an appropriate modification which makes it explicitly T-duality-invariant [6].

There is an analogous story for the unstable Dp -branes of Type II superstring theory which occur at the “wrong” values of p [28,29]. From the form of the corresponding open string scattering amplitudes, it was proposed that a system of N unstable branes wrapping a submanifold Σ of space–time generates a Ramond–Ramond coupling of the form [28–30]

$$\tilde{S}_{\text{WZ}}^{(0)\text{non}} = \int_{\Sigma} C \wedge d \text{tr}_N T e^{(F_A - B)/2\pi i} \wedge e^{d/2} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}, \tag{1.4}$$

where T is the Hermitian tachyon field which lives on the non-BPS branes and which belongs to the adjoint representation of the $U(N)$ gauge group. This form of the RR coupling has been shown to correctly reproduce, modulo the gravitational couplings, the charge formula for BPS D-branes after tachyon condensation. However, as in (1.3), one expects interactions between the gauge fields and higher powers of the tachyon field [14]. Based on this observation, it was proposed that the full Wess–Zumino action describing this coupling is given by [31]

$$\tilde{S}_{\text{WZ}}^{\text{non}} = \int_{\Sigma} C \wedge \sum_{n,m \geq 0} a_{nm} \text{tr}_N (D_A T)^{2n+1} T^{2m} \wedge e^{(F_A - B)/2\pi i} \wedge e^{d/2} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}, \tag{1.5}$$

where a_{nm} are undetermined numerical coefficients. The gauge-covariant derivative in (1.5) is given by $D_A T = dT + [A, T]$. Again, modulo the gravitational parts, action (1.5) correctly reproduces the Wess–Zumino term for BPS D-branes after tachyon condensation. It likewise admits a T-duality-invariant extension [6,32].

In what follows we shall show that this is indeed the case. We will find that the explicit realization of the proposed expansion (1.5) is given by an RR coupling of the form

$$\begin{aligned}
 S_{\text{WZ}}^{\text{non}} = \int_{\Sigma} C \wedge \text{tr}_N \left\{ \exp \left(\frac{1}{2\pi i} (T^2 + D_A T) + \sum_{r,s=1}^{\infty} \mathcal{E}_{rs}[A, T] \right) \right. \\
 \left. - \exp \left(\frac{1}{2\pi i} (T^2 - D_A T) + \sum_{r,s=1}^{\infty} \mathcal{E}_{rs}[A, -T] \right) \right\} \\
 \wedge e^{(F_A - B)/2\pi i} \wedge e^{d/2} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}}, \tag{1.6}
 \end{aligned}$$

where $\mathcal{E}_{rs}[A, T]$ are involved invariant functions of the gauge and tachyon fields on the system of non-BPS D-branes. They are given explicitly in (4.23). In fact, we will find that the lowest-order couplings (1.4) are induced by regarding the system of unstable D-branes as the naive dimensional reduction of a stable system in one higher dimension. Instead, the full action (1.6) comes from realizing the unstable branes in terms of a certain projection of a brane–antibrane system [28,33], and hence from the superconnection formalism that is appropriate to such systems [5,6,15]. In this sense, formula (1.4) may be thought of as a sort of “field theoretical” limit of the “stringy” action (1.6).

While the boundary string field formalism yields explicit forms for the Ramond–Ramond couplings of unstable systems of D-branes that give the correct brane tensions in processes involving tachyon condensation [5,6], we will take a more mathematical approach to the construction of the effective actions for these systems. The present approach focuses more closely on a set of generalized Dirac operators associated to the unstable D-branes, which yield an equivalent description of their geometry as that by superconnections and which serve as a probe of the low-energy open string dynamics. With this analysis, we will develop an intuitive, geometric understanding of the role of the tachyon field on non-BPS D-branes, and hence to the origins of their worldvolume effective field theories. A real virtue of the present formalism is that it enables the construction of *global* expressions for the worldvolume actions. In addition to yielding mathematical constructions of the Ramond–Ramond couplings, the Dirac operators yield very compact, spectral forms for the kinetic parts of the effective action describing the propagation of the worldvolume fields. Such generalized Dirac–Born–Infeld type actions have been previously proposed in [15,34,35]. The conditions required for tachyon condensation are thereby reproduced in a very natural way. We will show that the RR couplings for such generalized vortex configurations reduce to the anticipated forms for supersymmetric D-branes, generalizing earlier calculations to incorporate the worldvolume gravitational couplings. We will also describe the modifications of these actions required for T-duality invariance, although at present we do not have a natural geometric origin for these extra couplings.

Although for most of our analysis we restrict attention to D-branes in Type II superstring theory for simplicity, the techniques developed can be generalized to other string theories and to other brane systems (see [16] for further examples). In particular, a system of stable, supersymmetric D-branes is also a special instance of this general formalism. The formalism of generalized Dirac operators then suggests another intriguing relationship between Ramond–Ramond charges and K-theory. More precisely, it leads immediately to an interpretation of D-brane charge in terms of bivariant K-theory, and hence analytic K-homology. The relationship unveiled here complements previous considerations which suggest that D-brane charge should really be associated to K-homology [36,37]. As we will show, the formalism of analytic K-homology takes a particularly transparent form when applied to systems of unstable D-branes.

The structure of the remainder of this paper is as follows. In Section 2, we will give a mathematical introduction to the theory of superconnections and how they naturally describe the geometry of brane–antibrane systems. We also introduce the generalized Dirac operators which will play a prominent role in this paper, and also how the standard formalism for dealing with gauge anomalies can be extended to this case. In Section 3, we derive the form of the Ramond–Ramond couplings on brane–antibrane systems by using index theoretical arguments based on the identification of gauge anomalies generalized to the case of superconnection gauge fields. We describe the modifications of these actions due to non-abelian structure groups and topologically trivial B -fields. We also describe how tachyon condensation processes emerge naturally within this geometrical formalism, and how the standard anomalous couplings on systems of stable D-branes are reproduced from these RR couplings via the global bound-state construction. In Section 4, we then turn to the derivation of the Ramond–Ramond couplings on systems of unstable D-branes. We show how the leading-order terms in the tachyon field arise from a dimensional reduction of a stable system in one higher dimension. We then relate the couplings to the formalism of superconnections by deriving the full expansion in powers of the tachyon field via reduction from a brane–antibrane system. The resulting actions are also shown to reduce appropriately for Higgs profiles of the tachyon field. In Section 5, we indicate how the couplings universally extend to all branes of Type II superstring theory and M-theory, and hence illustrate the power of the analysis in that it can be applied to many other string theories and brane systems than the ones considered in this paper. Finally, in Section 6, we investigate the intimate relationship between D-brane charges and K-theory in light of the present analysis. We show that the formalism of superconnections and generalized Dirac operators leads naturally to the interpretation of D-brane charges in terms of Fredholm modules and bivariant K-theory, and hence analytic K-homology. We also show how the reductions leading to the charge formulas for unstable D-branes admit natural interpretations within this homological framework, thereby lending further support to the suggestion that D-brane charge should be properly understood within the context of K-homology.

2. Superconnections on brane–antibrane systems

In this section, we will describe the geometry of brane–antibrane systems from mostly a mathematical perspective, primarily to introduce notions that will play a fundamental role

in later sections. Such systems are \mathbf{Z}_2 -graded objects [9] and are thereby most naturally described using the language of superconnections [11,12] (see [13] for a concise introduction). Superconnections were originally introduced as geometric objects associated with graded vector bundles whereby the conventional integer grading by differential form degree is replaced with a \mathbf{Z}_2 -grading, giving more freedom to the standard constructions of differential geometry. In [11], the Chern–Weil invariants of a superbundle were constructed and the definition of the Chern character of a superconnection was given. We will focus primarily on the basic result that a certain class of superconnections are in a one-to-one correspondence with (generalized) Dirac operators, a fact that will be at the heart of the analysis of this paper. Earlier descriptions of the geometry of Higgs fields using the formalism of superconnections can be found in [38].

2.1. Chan–Paton superbundles

Consider a system of N^+ coincident D p -branes and N^- D \bar{p} -branes in Type II superstring theory.¹ We assume that the branes all wrap a common worldvolume Σ of dimension $p+1$ in the 10-dimensional space–time manifold X which is endowed with a spin structure and a Riemannian metric. We are interested in the properties of Chan–Paton gauge bundles $E \rightarrow \Sigma$ over the brane–antibrane worldvolume. The open string Hilbert space of the brane–antibrane system has a natural \mathbf{Z}_2 -grading which may be associated to it [9]. The Chan–Paton gauge group of the p – \bar{p} pairs is $U(N^+ + N^-)$ and it acts on the Hilbert space $\mathcal{H} = \mathbf{C}^{N^+ + N^-}$. It has an index $a = +$ for an open string ending on a p -brane and $a = -$ for an open string ending on a \bar{p} -brane. The endpoints of the p – \bar{p} open strings therefore carry a charge which takes values in a graded quantum Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. We may regard the $a = +$ state as bosonic and $a = -$ as fermionic.

This implies that any complex vector bundle $E \rightarrow \Sigma$ inherits this \mathbf{Z}_2 -grading and becomes a superbundle $E = E^+ \oplus E^-$, i.e. a bundle whose fibres $E_x = E_x^+ \oplus E_x^-$, $x \in \Sigma$, are graded complex vector spaces. The bundles E^+ and E^- are identified with the $U(N^+)$ and $U(N^-)$ Chan–Paton gauge bundles on the branes and antibranes, respectively. When $N^+ = N^-$, E^+ and E^- are topologically the same so that when such a collection of branes annihilates to the vacuum state there is no overall D-brane charge [2]. For $N^+ \neq N^-$, after brane–antibrane annihilation one is left with an excited state that has the Ramond–Ramond charge of $N^+ - N^-$ D-branes.

This means that the natural geometrical objects to consider on the p – \bar{p} worldvolume are not ordinary gauge connections, but rather superconnections [11–13]. For this, we let $\Omega(\Sigma) = \bigoplus_{k \geq 0} \Omega^k(\Sigma)$ be the graded algebra of smooth complex-valued differential forms over Σ with \mathbf{Z} -grading defined by the form degree k . The space $\Omega(\Sigma, E)$ of smooth E -valued differential forms on Σ then has a natural $\mathbf{Z} \times \mathbf{Z}_2$ -grading, but we will be mainly concerned with its total \mathbf{Z}_2 -grading defined by $\Omega(\Sigma, E) = \Omega^+(\Sigma, E) \oplus \Omega^-(\Sigma, E)$, where

$$\Omega^\pm(\Sigma, E) = \bigoplus_{k \geq 0} (\Omega^{2k}(\Sigma, E^\pm) \oplus \Omega^{2k+1}(\Sigma, E^\mp)). \quad (2.1)$$

¹ For $p = 9$, tadpole anomaly cancellation requires there to be an equal number of space–time filling branes and antibranes in the Type IIB vacuum state and $N^+ = N^-$.

A superconnection is then any odd linear operator \mathbb{A} on the $\Omega(\Sigma)$ -module $\Omega(\Sigma, E)$, i.e. $\mathbb{A} : \Omega^\pm(\Sigma, E) \rightarrow \Omega^\mp(\Sigma, E)$, that satisfies the Leibnitz rule

$$[\mathbb{A}, \beta]^+ = d\beta, \quad \beta \in \Omega(\Sigma), \tag{2.2}$$

where $[\cdot, \cdot]^+$ denotes the graded commutator. Note that a superconnection does not necessarily send k -forms to $(k + 1)$ -forms, but rather odd (resp. even) elements to even (resp. odd) elements. Nonetheless, because of the Leibnitz property (2.2), the superconnections on a Chan–Paton superbundle always form an affine space modelled on some set of local operators.

If ∇ is an arbitrary connection, then, by the Leibnitz rule, $\mathbb{A} - \nabla$ commutes with elements of $\Omega(\Sigma)$, and so it can be represented by the exterior product with an odd matrix-valued form \mathcal{A} , i.e. $(\mathbb{A} - \nabla)\beta = \mathcal{A} \wedge \beta$ for some $\mathcal{A} \in \Omega^-(\Sigma, \text{End } E)$. Thus any superconnection can be written in terms of a fixed, fiducial connection ∇ as

$$\mathbb{A} = \nabla + \mathcal{A}. \tag{2.3}$$

From the tensor product grading on the endomorphism algebra $\Omega(\Sigma, \text{End } E)$, we have

$$\begin{aligned} \Omega^\pm(\Sigma, \text{End } E) &= [\Omega^\mp(\Sigma) \otimes (\text{Hom}(E^+, E^-) \oplus \text{Hom}(E^-, E^+))] \\ &\oplus [\Omega^\pm(\Sigma) \otimes (\text{End } E^+ \oplus \text{End } E^-)] \end{aligned} \tag{2.4}$$

with the multiplication on $\Omega(\Sigma, \text{End } E) = \Omega^+(\Sigma, \text{End } E) \oplus \Omega^-(\Sigma, \text{End } E)$ given by

$$(\alpha \otimes a) \cdot (\beta \otimes b) = (-1)^{|a||\beta|}(\alpha \wedge \beta) \otimes (a \circ b) \tag{2.5}$$

for $\alpha, \beta \in \Omega(\Sigma)$ and $a, b \in \text{End } E$, where $|a|$ denotes the total degree of a . With respect to this \mathbf{Z}_2 -grading, we may decompose the linear operator \mathcal{A} as

$$\mathcal{A} = \begin{pmatrix} A^+ & T^- \\ T^+ & A^- \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} A^\pm &= \sum_{k \geq 0} A_{(2k+1)}^\pm \in \Omega^-(\Sigma) \otimes \text{End } E^\pm, \\ T^\pm &= \sum_{k \geq 0} T_{(2k)}^\pm \in \Omega^+(\Sigma) \otimes \text{Hom}(E^\pm, E^\mp). \end{aligned} \tag{2.7}$$

The one-form components of A^\pm define ordinary gauge connections $A_{(1)}^\pm$ on the bundles E^\pm . The zero-form components of T^\pm are odd matrix-valued bundle maps $T_{(0)}^\pm : C^\infty(\Sigma, E^\pm) \rightarrow C^\infty(\Sigma, E^\mp)$ with $(T_{(0)}^\pm)^\dagger = T_{(0)}^\mp$, i.e. smooth sections of the product bundles $E^\mp \otimes (E^\pm)^*$, where $(E^\pm)^*$ is the dual of E^\pm . They define the complex scalar tachyon fields $T(x) = T_{(0)}^+(x)$ of the brane–antibrane system. At a point $x \in \Sigma$, the tachyon field is a linear fibre map $T(x) : E_x^+ \rightarrow E_x^-$ and its adjoint $T^\dagger(x) : E_x^- \rightarrow E_x^+$.

If θ is any matrix-valued form, then the Leibnitz rule and the Jacobi identity for the graded commutator imply that

$$[[\mathbb{A}, \theta]^+, \beta]^+ = [\mathbb{A}, [\theta, \beta]^+]^+ + (-1)^{|\theta||\beta|}[d\beta, \theta]^+ \quad \forall \beta \in \Omega(\Sigma). \tag{2.8}$$

Eq. (2.8) defines the action of the covariant derivative $\mathbb{A}\theta$ in $\Omega(\Sigma, \text{End } E)$, i.e. $\mathbb{A}\theta$ is identified as multiplication by the operator $[\mathbb{A}, \theta]^+$. Since \mathbb{A} is odd, we have $[\mathbb{A}, \mathbb{A}]^+ = 2\mathbb{A}^2$, and so from the Leibnitz rule and (2.8), we find

$$[\mathbb{A}^2, \beta]^+ = [\mathbb{A}, [\mathbb{A}, \beta]^+]^+ = d(d\beta) = 0 \quad \forall \beta \in \Omega(\Sigma). \tag{2.9}$$

It follows that \mathbb{A}^2 lives in the endomorphism bundle $\Omega^+(\Sigma, \text{End } E)$. The operator $F_{\mathbb{A}} = \mathbb{A}^2$ is the curvature of the superconnection \mathbb{A} and it satisfies the Bianchi identity

$$\mathbb{A}F_{\mathbb{A}} = [\mathbb{A}, F_{\mathbb{A}}]^+ = [\mathbb{A}, \mathbb{A}^2]^+ = 0. \tag{2.10}$$

The curvature is in general a sum

$$F_{\mathbb{A}} = \sum_{k \geq 0} \mathcal{F}_{(k)}, \quad \mathcal{F}_{(k)} \in \Omega^k(\Sigma, \text{End } E). \tag{2.11}$$

By using the grading (2.4), the first few components are found to be

$$\mathcal{F}_{(0)} = \begin{pmatrix} T^\dagger T & 0 \\ 0 & TT^\dagger \end{pmatrix}, \tag{2.12}$$

$$\mathcal{F}_{(1)} = \begin{pmatrix} 0 & DT^\dagger \\ DT & 0 \end{pmatrix}, \tag{2.13}$$

$$\mathcal{F}_{(2)} = \begin{pmatrix} R_\nabla + F^+ & 0 \\ 0 & R_\nabla + F^- \end{pmatrix}, \tag{2.14}$$

etc., where $R_\nabla = \nabla^2$ is the curvature of the fiducial derivation ∇ ,

$$F^\pm = F_{A^\pm} = \nabla A_{(1)}^\pm + A_{(1)}^\pm \wedge A_{(1)}^\pm \tag{2.15}$$

are the curvatures of the gauge connections $A_{(1)}^\pm$ on E^\pm , and

$$DT = \nabla T + A_{(1)}^- T + T A_{(1)}^+, \quad DT^\dagger = \nabla T^\dagger + A_{(1)}^+ T^\dagger + T^\dagger A_{(1)}^-. \tag{2.16}$$

The curvature component (2.12) defines a term bilinear in the tachyon field, the component (2.13) defines a kinetic term in terms of covariant derivatives of the tachyon field, and (2.14) yields the usual Yang–Mills field strengths on the branes and antibranes in the absence of the tachyon field (and all other higher rank fields).

Everything we have said thus far has been rather general, and we now need to input some more physical requirements. We will focus on the low-lying excitations of the brane–antibrane system, which are described by the p - p , p - \bar{p} and \bar{p} - \bar{p} open string states. The p - p (resp. \bar{p} - \bar{p}) open string spectrum consists of a massless $U(N^+)$ (resp. $U(N^-)$) supersymmetric Yang–Mills multiplet, along with massive excitations. In these bosonic components, the NS sector tachyon state is removed by the standard GSO projection. The open string wavefunctions are the products $\psi = \psi_{\text{osc}} \otimes \psi_{\text{CP}}$ of the usual mode decompositions and the Chan–Paton factors $\psi_{\text{CP}} \in U(N^+ + N^-)$. The GSO projection operator is

$$P_{\text{GSO}} = \frac{1}{2}(\mathbb{1} + (-1)^F), \tag{2.17}$$

where F is the worldsheet fermion number operator. Its action on the Chan–Paton factors may be represented as

$$(-1)^F : \psi_{\text{CP}} \mapsto \varepsilon \psi_{\text{CP}} \varepsilon \tag{2.18}$$

in terms of the usual grading automorphism ε for the superbundle E ,

$$\varepsilon = \begin{pmatrix} \mathbb{1}_{N^+} & 0 \\ 0 & -\mathbb{1}_{N^-} \end{pmatrix}, \tag{2.19}$$

such that $E = E^+ \oplus E^-$ decomposes into the ± 1 eigenspaces of ε . With respect to this decomposition, the p – p and \bar{p} – \bar{p} open strings have diagonal Chan–Paton wavefunctions ψ_{CP} which are even under action (2.18) of $(-1)^F$, leading to the usual GSO projection on the oscillator modes ψ_{osc} . On the other hand, the Chan–Paton wavefunctions for the p – \bar{p} and \bar{p} – p open string states are off-diagonal and odd under $(-1)^F$, leading to a reversed GSO projection on the corresponding oscillators. This means that in these sectors, the massless $U(N^+)$ and $U(N^-)$ vector supermultiplets are projected out, and the tachyon survives [39,40]. These features of the low-energy open string theory are already encoded in the first two components of the superconnection constructed above. But it also implies that, as far as the low-energy effective field theory on the brane–antibrane worldvolume is concerned, all higher form degree components of \mathcal{A} in (2.6) are absent, because the GSO projection (2.17) eliminates the off-diagonal fermionic gauge fields. The superconnections relevant to the low-energy physics of brane–antibrane systems are therefore precisely of the type considered originally in [11], and henceforth we shall thereby deal only with the superconnection defined by taking $A^\pm = A_{(1)}^\pm$ and $T^\pm = T_{(0)}^\pm$ in (2.6). Then, the superconnection field strength (2.11) reduces to

$$F_{\mathbb{A}} = \begin{pmatrix} R_{\nabla} + F^+ + T^\dagger T & DT^\dagger \\ DT & R_{\nabla} + F^- + TT^\dagger \end{pmatrix}. \tag{2.20}$$

The stated properties of the GSO projection also imply that the gauge symmetry group G of the superbundle $E = E^+ \oplus E^-$, which is generically the unitary Lie supergroup $U(N^+|N^-)$, is instead that which is lifted from the structure groups of the Chan–Paton bundles E^\pm over the branes and antibranes, i.e. $G = U(N^+) \times U(N^-)$.² With $g = g^+ \oplus g^-$ an automorphism of the associated principal bundle, the gauge transformation law of the superconnection is given as

$$\mathcal{A} \mapsto g \mathcal{A} g^{-1} + g \nabla g^{-1}, \quad F_{\mathbb{A}} \mapsto g F_{\mathbb{A}} g^{-1}, \tag{2.21}$$

which is equivalent to the component transformation rules

$$\begin{aligned} A_{(1)}^\pm &\mapsto g^\pm A_{(1)}^\pm (g^\pm)^{-1} + g^\pm \nabla (g^\pm)^{-1}, & F^\pm &\mapsto g^\pm F^\pm (g^\pm)^{-1}, \\ T &\mapsto g^- T (g^+)^{-1}, & T^\dagger &\mapsto g^+ T^\dagger (g^-)^{-1}. \end{aligned} \tag{2.22}$$

In other words, the component gauge fields transform in the usual way under the adjoint actions of $U(N^+)$ and $U(N^-)$, while the tachyon field transforms in a bi-fundamental

² An alternative interpretation of the $U(N^+|N^-)$ brane–antibrane supergroup symmetry in the context of topological string theory is given in [41].

unitary group representation $\overline{\mathbf{N}^-} \otimes \mathbf{N}^+$, or equivalently it carries charges $(\overline{\mathbf{N}^-}, \mathbf{N}^+)$ with respect to the $U(N^-) \times U(N^+)$ brane–antibrane gauge fields $(A_{(1)}^-, A_{(1)}^+)$. But within the more general superconnection formalism presented above, we see that the tachyon field may actually be regarded as a sort of generalized gauge field on the discrete space \mathbf{Z}_2 , i.e. we may regard the brane–antibrane worldvolume as the two-sheeted manifold $\Sigma \times \mathbf{Z}_2$, whose two sheets are connected together by T .

2.2. Spin geometry

The properties of the GSO projection on the low-lying excitations of the brane–antibrane system have in addition some important consequences for the spin geometry of the world-volume manifold Σ . We work in Type IIB superstring theory so that $\dim \Sigma = p + 1$ is even, and assume that Σ is oriented with a given Riemannian structure. The corresponding result for Type IIA D-branes will follow by T-duality. Let $C\ell(\Sigma)$ be the complex Clifford bundle over Σ whose fibre over a point $x \in \Sigma$ is the complexified Clifford algebra $C\ell(T_x^* \Sigma)$ with respect to a Hermitian structure $\langle \cdot, \cdot \rangle$ on the fibres of the cotangent bundle $T^* \Sigma$. The smooth sections of the Clifford bundle form the algebra $\mathcal{C} = C^\infty(\Sigma, C\ell \Sigma)$. Let $S \rightarrow \Sigma$ be a spinor bundle of rank $2^{(p+1)/2}$, and let $c : T^* \Sigma \rightarrow \text{End } S$ be a spin^c-structure on the worldvolume Σ . This requires, in addition to the orientability of Σ , that its second Stiefel–Whitney class $w_2(\Sigma)$ be the mod 2 reduction of an integral cohomology class. The linear bundle map c satisfies

$$c(v)^2 + \langle v, v \rangle = 0, \quad v \in T^* \Sigma, \tag{2.23}$$

and it can be universally extended to an irreducible Clifford action on S , i.e. c extends uniquely to an algebra isomorphism $c : C\ell(\Sigma) \xrightarrow{\cong} \text{End } S$ which is compatible with the property (2.23). In particular, the algebra \mathcal{C} acts irreducibly on the space $C^\infty(\Sigma, S)$ of smooth sections of the spinor bundle so that $\mathcal{C} \cong \text{End } S$.

The Clifford bundle is a superbundle $C\ell(\Sigma) = C\ell^+(\Sigma) \oplus C\ell^-(\Sigma)$ with grading automorphism $v \mapsto -v$. The spinor bundle is also a superbundle $S = S^+ \oplus S^-$ with \mathbf{Z}_2 -grading defined as follows. For each $x \in \Sigma$, let θ^a be an oriented orthonormal frame in $T_x^* \Sigma$, and set

$$\gamma^a = c(\theta^a). \tag{2.24}$$

The γ 's generate locally the Euclidean–Dirac algebra

$$\gamma^a \gamma^b + \gamma^b \gamma^a = -2\delta^{ab}, \quad a, b = 1, \dots, p + 1. \tag{2.25}$$

The Hermitian chirality operator $\gamma_c = i^{(p+1)/2} \gamma^1 \gamma^2 \dots \gamma^{p+1}$ then satisfies

$$(\gamma_c)^2 = \mathbb{1}_{2^{(p+1)/2}}, \quad \gamma_c \gamma^a = -\gamma^a \gamma_c. \tag{2.26}$$

The sub-bundles S^\pm are taken to be the ± 1 eigenspaces of the chirality operator γ_c . They have the same rank $2^{(p-1)/2}$, and because of (2.26), the (left) action of $C\ell(\Sigma)$ on S , defined by $a \cdot s = c(a)s$, preserves this \mathbf{Z}_2 -grading, i.e. $C\ell^+(\Sigma) \cdot S^\pm \subset S^\pm$ and $C\ell^-(\Sigma) \cdot S^\pm \subset S^\mp$. Thus the superbundle $S = S^+ \oplus S^-$ is a graded (left) Clifford module.

We will now incorporate the graded Chan–Paton vector bundle $E = E^+ \oplus E^-$ of the brane–antibrane system above. For this, we introduce the twisted spinor bundle

$$S_E = S \otimes E = S_E^+ \oplus S_E^- \tag{2.27}$$

with Clifford action $c \otimes \mathbb{1}$, which we also denote by c . From the properties of the GSO projection described in the previous section, it follows that the only chiral spinors that appear in the spectrum of the brane–antibrane system are those which are lifted over the bundles E^\pm . The GSO projection removes the massless fermionic modes in the open string $p-\bar{p}$ and $\bar{p}-p$ sectors. This implies that the \mathbf{Z}_2 -grading of (2.27) is just that which arises from the induced tensor product grading of the two superbundles S and E ,

$$S_E^\pm = (S^\pm \otimes E^+) \oplus (S^\mp \otimes E^-). \tag{2.28}$$

This makes S_E a Clifford module. It also implies that the appropriate superconnection on (2.27) is a Clifford superconnection \mathbb{S} , i.e. \mathbb{S} respects the \mathcal{C} -module structure by satisfying a second Leibnitz rule (in addition to (2.2)) involving the Clifford action

$$[\mathbb{S}, c(\beta)]^+ = c(\nabla\beta) \quad \forall \beta \in \Omega(\Sigma). \tag{2.29}$$

Here $\nabla : C^\infty(\Sigma, T\Sigma) \rightarrow C^\infty(\Sigma, T^*\Sigma \otimes T\Sigma)$ is the Levi–Civita connection on the tangent bundle $T\Sigma$ which can be written locally in terms of Christoffel symbols $\Gamma \in \Omega^1(\Sigma, \text{so}(T^*\Sigma))$ as $\nabla = d + \Gamma$.

The spin connection ∇^s on the spinor bundle has the property (2.29). Because of the canonical bundle isomorphism $\Omega(\Sigma) \cong C\ell(\Sigma)$, it is a map $\nabla^s : C^\infty(\Sigma, S) \rightarrow C^\infty(\Sigma, T^*\Sigma \otimes S)$ which can be written locally as

$$\nabla^s = d + \omega(\Gamma), \tag{2.30}$$

where $\omega : \text{so}(T^*\Sigma) \rightarrow C\ell(\Sigma)$ is the spinor representation of the Lie algebra of the orthogonal group with

$$\omega(\Gamma_i) = -\frac{1}{4}\Gamma_{ia}^b \gamma^a \gamma^b. \tag{2.31}$$

The corresponding curvature is

$$(\nabla^s)^2 = \omega(R_\nabla) = \frac{1}{4}R_{baij} \gamma^a \gamma^b dx^i \wedge dx^j, \tag{2.32}$$

where $R_\nabla = d\Gamma + \Gamma \wedge \Gamma \in \Omega^2(\Sigma, \text{so}(T^*\Sigma))$ is the Riemann curvature tensor. It follows that if \mathbb{S} is any Clifford superconnection on S_E , then $\mathbb{S} - \nabla^s \otimes \mathbb{1}$ commutes with the Clifford action c , and therefore the most general Clifford superconnection is of the form

$$\mathbb{S} = \nabla^s \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{A}, \tag{2.33}$$

where \mathbb{A} is any superconnection on the twisting bundle E . Since $[\nabla^s \otimes \mathbb{1}, \mathbb{1} \otimes \mathbb{A}] = 0$, the curvature of (2.33) splits as

$$\mathbb{S}^2 = \omega(R_\nabla) \otimes \mathbb{1} + \mathbb{1} \otimes F_\mathbb{A} \tag{2.34}$$

into the sum of a purely gravitational piece and an internal gauge piece. Thus the Clifford superconnections \mathbb{S} will naturally incorporate both gauge and gravitational effects into the ensuing computations. We note that (2.34) is compatible with the identification $F_{\mathbb{A}} \in \Omega^+(\Sigma, \text{End } E)$, because the Leibnitz rule (2.29) implies

$$[\mathbb{S}^2, c(\beta)]^+ = [\mathbb{S}, [\mathbb{S}, c(\beta)]^+]^+ = c(\nabla^2 \beta) = c(R_{\nabla} \wedge \beta), \tag{2.35}$$

and also $[\omega(R_{\nabla}), c(\beta)]^+ = c(R_{\nabla} \wedge \beta)$. This shows that $F_{\mathbb{A}} = \mathbb{S}^2 - \omega(R_{\nabla})$ commutes with all $c(\beta)$, as expected.

Thus, given any superconnection \mathbb{A} on a Chan–Paton superbundle $E = E^+ \oplus E^-$, there is a natural extension to a Clifford superconnection (2.33) on the twisted spinor bundle (2.27). In some applications, such as the bound-state constructions of lower-dimensional D-branes from the brane–antibrane system, we will require that the Chan–Paton bundle E itself define a Clifford module. Then Schur’s lemma implies that any map $C^\infty(\Sigma, S) \rightarrow C^\infty(\Sigma, E)$ which commutes with the Clifford action c is of the form $\psi \mapsto \psi \otimes w$, where $w \in C^\infty(\Sigma, W)$ with $W = \text{Hom}_{C\ell\Sigma}(S, E)$ the bundle of intertwining maps [13]. Moreover, any endomorphism of $C^\infty(\Sigma, E)$ that commutes with the Clifford action is of the form $\psi \otimes w \mapsto \psi \otimes Lw$ for some bundle map $L : W \rightarrow W$ so that $\text{End}_{C\ell\Sigma} E \cong \mathbb{1} \otimes \text{End } W$. The entire matrix bundle $\text{End } E$ is then generated by the sub-bundle $C\ell(\Sigma) \cong \text{End } S \otimes \mathbb{1}$ acting by the spinor representation, and by its commutant $\mathbb{1} \otimes \text{End } W$ so that

$$\text{End } E \cong C\ell(\Sigma) \otimes \text{End } W. \tag{2.36}$$

This means that any Clifford module $C^\infty(\Sigma, E)$ on a spin^c -manifold Σ comes from a twisted spinor bundle $E \cong S_W = S \otimes W$ with the canonical grading defined by (2.28). A generic Clifford superconnection \mathbb{A} on E may then be written as $\mathbb{A} = \nabla^s \otimes \mathbb{1} + \mathbb{1} \otimes S_W$, where S_W is any superconnection on the intertwining bundle W .

2.3. Generalized Dirac operators

We will now describe some general properties of Dirac operators associated with superconnections that are compatible with the Clifford action, with respect to a chosen spin^c -structure on Σ . These operators will play a fundamental role in all considerations of this paper. Let \mathbb{S} be such a Clifford superconnection, as in (2.33). We identify the algebra $\mathcal{C} = C^\infty(\Sigma, C\ell\Sigma)$ with the algebra of differential forms $\Omega(\Sigma)$ by the symbol map isomorphism $\sigma(\beta) = c(\beta)\mathbb{1}_S$. The inverse “quantization” map $Q : \Omega(\Sigma) \xrightarrow{\sim} C^\infty(\Sigma, C\ell\Sigma)$ then allows one to represent exterior products of forms by Clifford multiplication [13]. We define the associated twisted Dirac operator by the following composition of maps:

$$\mathbb{D} : C^\infty(\Sigma, S_E) \xrightarrow{\mathbb{S}} C^\infty(\Sigma, T^*\Sigma \otimes S_E) \xrightarrow{c} C^\infty(\Sigma, S_E). \tag{2.37}$$

In local coordinates where $\mathbb{S} = \theta^a \otimes \mathbb{S}_a$ we have $\mathbb{D} = \gamma^a \mathbb{S}_a$, and with respect to the grading (2.4), we may write

$$\mathbb{D} = \begin{pmatrix} \mathbb{D}_A^+ & \mathcal{G}_s \otimes T^\dagger \\ \mathcal{G}_s \otimes T & \mathbb{D}_A^- \end{pmatrix}, \tag{2.38}$$

where

$$\mathcal{D}_A^\pm = Q(\nabla^s \otimes \mathbb{1} + \mathbb{1} \otimes A_{(1)}^\pm) = \gamma^a \nabla_a^s \otimes \mathbb{1} + \gamma^a \otimes A_{(1)a}^\pm. \tag{2.39}$$

Here $\mathcal{G}_s = \mathcal{G}_s^\dagger$ is a constant operator on $C^\infty(\Sigma, S)$ which we have combined with the tachyon field in the zero-form component of the superconnection \mathbb{A} .

The Dirac operator \mathcal{D} on $C^\infty(\Sigma, S_E)$ satisfies three fundamental conditions, namely

- It is an odd operator, i.e. $\mathcal{D} : C^\infty(\Sigma, S_E^\pm) \rightarrow C^\infty(\Sigma, S_E^\mp)$.
- It respects the \mathcal{C} -module structure of $C^\infty(\Sigma, S_E)$, i.e. \mathcal{D} satisfies (2.29).
- It transforms homogeneously under local gauge transformations (2.21) and (2.22), i.e. $\mathcal{D} \mapsto g\mathcal{D}g^{-1}$.

In particular, there is a one-to-one correspondence between Dirac operators which are compatible with a given Clifford action and Clifford superconnections [13] so that \mathcal{D} and \mathbb{S} carry the same information. In this sense, superconnections may be thought of as quantizations of ordinary connections [11].

2.4. Transgression and descent equations

Most of our subsequent analysis will rely on the computation of gauge anomalies associated with the brane–antibrane system. In this section, we will show how the standard formalism for Yang–Mills anomalies carries over to the case of superconnection gauge fields. For this, we consider an arbitrary function $I(\mathcal{A}, F_{\mathbb{A}})$ of a superconnection \mathbb{A} and its curvature $F_{\mathbb{A}}$. We introduce the integral operator

$$\mathbb{K}I(\mathcal{A}, F_{\mathbb{A}}) = \int_0^1 dt \mathbb{K}_{(t)}I(t\mathcal{A}, F_{\mathbb{A}}(t)), \tag{2.40}$$

where $F_{\mathbb{A}}(t) = t d\mathcal{A} + t^2 \mathcal{A} \wedge \mathcal{A}$ is the curvature associated with $t\mathcal{A}$ which is a path in field space connecting the superconnection gauge fields 0 and \mathcal{A} , and the anti-differential operator $\mathbb{K}_{(t)}$ is defined by $\mathbb{K}_{(t)}(t\mathcal{A}) = 0$, $\mathbb{K}_{(t)}F_{\mathbb{A}}(t) = t\mathcal{A}$. From this definition, we can infer the generalization of the Cartan homotopy formula

$$I(\mathcal{A}, F_{\mathbb{A}}) = (\mathbb{K}d + d\mathbb{K})I(\mathcal{A}, F_{\mathbb{A}}). \tag{2.41}$$

We will now specialize to the case of the invariant polynomial $I(\mathcal{A}, F_{\mathbb{A}}) = \text{Tr}^+(F_{\mathbb{A}})^n$, where $n \geq 1$ and $\text{Tr}^+(\cdot) = \text{Tr}(\varepsilon \cdot)$ is the supertrace in the fundamental representation of the brane–antibrane gauge group $U(N^+) \times U(N^-)$. This polynomial is a closed form, because the Bianchi identity (2.10) implies

$$d \text{Tr}^+(F_{\mathbb{A}})^n = \text{Tr}^+(d(F_{\mathbb{A}})^n + [A^\wedge, (F_{\mathbb{A}})^n]^+) = 0. \tag{2.42}$$

The important property of this closed differential form is that its cohomology class is independent of the choice of superconnection \mathbb{A} [11]. To see this, let \mathbb{A}_s , $s \in \mathbf{R}$, be a one-parameter family of superconnections with curvatures $F_s = (\mathbb{A}_s)^2$. Then

$$\frac{\partial F_s}{\partial s} = \left[\mathbb{A}_s, \frac{\partial \mathbb{A}_s}{\partial s} \right]^+, \tag{2.43}$$

and applying this result to the invariant polynomial $\text{Tr}^+(F_s)^n$ gives

$$\begin{aligned} \frac{\partial \text{Tr}^+(F_s)^n}{\partial s} &= n \text{Tr}^+ \frac{\partial F_s}{\partial s} \wedge (F_s)^{n-1} = n \text{Tr}^+ \left[\mathbb{A}_s, \frac{\partial \mathbb{A}_s}{\partial s} (F_s)^{n-1} \right]^+ \\ &= d \text{Tr}^+ \frac{\partial \mathbb{A}_s}{\partial s} (F_s)^{n-1}, \end{aligned} \tag{2.44}$$

showing that any continuous deformation of the form $\text{Tr}^+(F_{\mathbb{A}})^n$ changes it by an exact form. In particular, the cohomology class determined by it is independent of the choice of profile for the tachyon field T , a fact which will be exploited throughout this paper.

From (2.42), we arrive at the generalized transgression formula [42]

$$\text{Tr}^+(F_{\mathbb{A}})^n = d\xi_{2n-1}^{(0)}(\mathcal{A}, F_{\mathbb{A}}), \tag{2.45}$$

where $\xi_{2n-1}^{(0)}$ is a generalized Chern–Simons form which using the homotopy formula (2.41) can be written as

$$\xi_{2n-1}^{(0)}(\mathcal{A}, F_{\mathbb{A}}) = \mathbb{K} \text{Tr}^+(F_{\mathbb{A}})^n = n \int_0^1 dt \text{Tr}^+ \mathcal{A} \wedge F_{\mathbb{A}}(t)^{n-1}. \tag{2.46}$$

The form (2.46) is the first member of the BRST complex constructed from the Lie algebra cohomology of $U(N^+) \times U(N^-)$. Let δ_{BRST} be the corresponding co-boundary operator which is associated with the infinitesimal gauge transformations (2.21). The BRST ghost field $\Lambda = \Lambda^+ \oplus \Lambda^-$ is the Cartan–Maurer one-form on the brane–antibrane gauge group, i.e. the tautological one-form

$$\Lambda = g^{-1} \delta_{\text{BRST}} g \tag{2.47}$$

at the identity element which sends a Lie algebra element onto itself. It satisfies the Cartan–Maurer equation

$$\delta_{\text{BRST}} \Lambda = -\Lambda \wedge \Lambda. \tag{2.48}$$

The other terms in the BRST complex are then obtained by replacing the superconnection via $\mathcal{A} \mapsto \mathcal{A} + \Lambda$ and with the curvature associated with the operator $d + \delta_{\text{BRST}}$. The appropriate generalizations of (2.46) then come from the paths of superconnections in field space which connect Λ with $\mathcal{A} + \Lambda$, and also 0 with Λ . They, respectively, read

$$\xi_{2n-1-k}^{(k)} = \begin{cases} n \int_0^1 dt \text{Tr}^+ [\mathcal{A} \wedge (F_{\mathbb{A}}(t) + (1-t) d\Lambda)^{n-1}]^{(k)} & \text{for } k = 0, 1, \dots, n-1, \\ n \int_0^1 dt \text{Tr}^+ [\Lambda \wedge (t d\Lambda + (t^2-t)\Lambda \wedge \Lambda)^{n-1}]^{(k)} & \text{for } k = n, n+1, \dots, 2n, \end{cases} \tag{2.49}$$

where the bracket $[\cdot]^{(k)}$ indicates to extract the terms of degree k in the BRST ghost field Λ .

The forms (2.49) are related to one another through the descent equations [43]

$$\delta_{\text{BRST}} \xi_{2n-1-k}^{(k)} = -d\xi_{2n-k-2}^{(k+1)}, \quad k = 0, 1, \dots, 2n-1. \tag{2.50}$$

The co-cycles $\xi_{2n-1-k}^{(k)}$ may also be obtained in a somewhat more geometric way via the Cheeger–Simons construction [44]. For this, we fix a superconnection on the graded universal bundle $EG \rightarrow BG$, where BG is a smooth classifying space of the brane–antibrane

gauge group $G = U(N^+) \times U(N^-)$. Let $\hat{\xi}_{2n-1-k}^{(k)}$ be a co-cycle representative of the Chern–Weil form associated to a Chern–Simons characteristic class in $H^{2n-1-k}(BG, \mathbf{R})$. Given a Chan–Paton superbundle $E \rightarrow \Sigma$ with superconnection \mathbb{A} , let $f : \Sigma \rightarrow BG$ be the map induced from a homotopy class of G -equivariant classifying maps on $E \rightarrow EG$. Then the Chern–Simons co-cycle of \mathbb{A} is given by the pullback

$$\xi_{2n-1-k}^{(k)} = f^* \hat{\xi}_{2n-1-k}^{(k)} \in H^{2n-1-k}(\Sigma, \mathbf{R}). \tag{2.51}$$

By suitably integrating the forms (2.51) over cycles of the worldvolume Σ , one obtains a family of co-cycles $\text{CS}_{2n-1-k}^{(k)}, \delta_{\text{BRST}} \text{CS}_{2n-1-k}^{(k)} = 0$, in the BRST cohomology $H^k(U(N^+) \times U(N^-), \mathbf{R})$ of the brane–antibrane gauge group. In what follows we shall deal mostly with the topological anomaly $\text{CS}_{2n-2}^{(1)} \in H^1(U(N^+) \times U(N^-), \mathbf{R})$ which corresponds to the $k = 1$ term in (2.50). The form

$$\xi_{2n-2}^{(1)} = n(n-1) \int_0^1 dt (1-t) \text{STr}^+ \Lambda \wedge d(\mathcal{A} \wedge F_{\mathbb{A}}(t)^{n-2}) \tag{2.52}$$

is then the solution to the Wess–Zumino consistency condition [43] for the chiral gauge anomaly in the brane–antibrane worldvolume field theory, where

$$\text{STr}(M_1, \dots, M_n) = \frac{1}{n!} \sum_{\pi \in S_n} \text{Tr}(M_{\pi_1} \cdots M_{\pi_n}) \tag{2.53}$$

is the symmetrized trace. For example, setting $n = 2$ in the above formulas yields the generalized Chern–Simons form

$$\begin{aligned} \xi_3^{(0)}(\mathcal{A}, F_{\mathbb{A}}) &= \text{Tr}(A_{(1)}^+ \wedge dA_{(1)}^+ + \frac{2}{3} A_{(1)}^+ \wedge A_{(1)}^+ \wedge A_{(1)}^+) \\ &\quad - \text{Tr}(A_{(1)}^- \wedge dA_{(1)}^- + \frac{2}{3} A_{(1)}^- \wedge A_{(1)}^- \wedge A_{(1)}^-) \\ &\quad + \text{Tr}(TDT^\dagger + T^\dagger DT). \end{aligned} \tag{2.54}$$

The first two terms in (2.54) are the standard three-dimensional Chern–Simons terms for the gauge fields on the branes and antibranes, respectively. The last term represents the modification of the ordinary anomaly formula due to the tachyon field.

The analysis of this section shows that the standard techniques for dealing with gauge anomalies carry through to the case of superconnections, with the appropriate modifications. It is also possible to incorporate modifications due to the gravitational terms on the brane–antibrane system by constructing superconnections from the generalized Dirac operators of the previous section. We shall return to this point in Section 6. Brane actions based on transgression forms have been constructed from a different point of view in [45].

3. Ramond–Ramond couplings on brane–antibrane systems

In this section, we will apply the previous considerations to a systematic derivation of the Ramond–Ramond couplings of brane–antibrane pairs. Amongst other objects, the brane–antibrane worldvolume Σ has spinor fields Ψ defined on it with kinetic term of the

form $\bar{\Psi} i\mathcal{D}\Psi$ in the total Lagrangian of the worldvolume field theory. If the fermion fields are chiral, then the functional integral over them yields the regularized determinant of the (generalized) Dirac operator \mathcal{D} . By incorporating supergravity couplings to the D-branes, the effective path integral measure for the brane–antibrane system will therefore contain the factor

$$\Theta = \det(i\mathcal{D}) e^{iZ}. \quad (3.1)$$

Here

$$Z = -\frac{\mu}{2} \int_{\Sigma} \mathcal{C} \wedge \mathcal{Y}, \quad (3.2)$$

where μ is a charge associated with the branes, $\mathcal{C} = \phi^*C = \sum_p \phi^*C_{(p)}$ the pullback to $\Sigma \hookrightarrow \phi X$ of the total RR potential,³ and the coupling \mathcal{Y} will be determined in what follows as an invariant polynomial of the gravitational and gauge curvatures on Σ . The determinant in (3.1) depends on the various bosonic fields which couple to Ψ and which live in a parameter space \mathcal{M} . In many instances, it is not a function on \mathcal{M} but rather a section of a complex line bundle $\mathcal{P}(\mathcal{D}) \rightarrow \mathcal{M}$ [46]. For chiral fermion fields, there is an obstruction to constructing a gauge-invariant regularized determinant of the Dirac operator \mathcal{D} . It is the topological chiral anomaly which is measured by the first Chern class of the determinant line bundle over \mathcal{M} . A two-form representative for this Chern class can be constructed as the transgression of the one-form $\text{CS}_{2n-2}^{(1)}$. If \mathcal{H}^+ denotes the positive energy spectral subspace of $i\mathcal{D}$, then the cohomology class $[\mathcal{H}^+] \in H^{2n-1}(\mathcal{M}, \mathbf{Z})$ measures the corresponding obstruction. The chiral gauge anomaly is the obstruction to constructing a corresponding global, non-zero trivializing section on $\mathcal{M} \rightarrow \mathcal{P}(\mathcal{D})$ which would allow the remaining functional integration over \mathcal{M} to be carried out in the brane–antibrane quantum field theory. The topology of the determinant line bundle $\mathcal{P}(\mathcal{D})$ is given by the Atiyah–Singer index theorem [47], which when expressed in differential geometric terms gives explicit forms for the chiral anomaly [48].

On the other hand, action (3.2) is not in general invariant under local gauge transformations of the Ramond–Ramond tensor potentials. Its exponential in (3.1) is therefore not a function on $\mathcal{M} \rightarrow \mathbf{S}^1$, but rather a unit norm section of a complex line bundle over the parameter space \mathcal{M} which is not necessarily covariantly constant. To make the brane–antibrane worldvolume quantum field theory well-defined, we will demand that the product of sections in (3.1) be a globally well-defined function on \mathcal{M} , i.e. that e^{-iZ} be a trivializing section for $\det i\mathcal{D}$. This will uniquely fix the form of the Ramond–Ramond coupling (3.2) to the brane–antibrane system. As in the previous section, we shall throughout assume that the worldvolume manifold Σ admits a spin^c -structure, which is equivalent to a certain topological constraint on the supergravity background that we shall discuss later on. This is the case for D-branes which wrap supersymmetric cycles in all Type II compactifications with vanishing cosmological constant [49].

³ We will assume throughout that the worldvolume Σ is embedded in the space–time X . While this requirement is not completely necessary, it will simplify some of the calculations in what follows.

3.1. Anomalous couplings

The perturbative, chiral gauge anomaly on the brane–antibrane worldvolume may be computed as the index of an appropriate Dirac operator [48]. For this, we first need to identify the massless spectrum in the brane–antibrane worldvolume field theory. As we have mentioned, after the GSO projection, there remains chiral fermion zero modes in the $p-p$ and $\bar{p}-\bar{p}$ open string sectors, but not in the $p-\bar{p}$ sectors. In these latter sectors there are of course the superpartners of the tachyon fields, which are massless bi-fundamental fermion fields of opposite chirality. The fermions are all in a one-to-one correspondence with the relevant open string Ramond sector ground states. Open string quantization requires them to be sections of the spinor bundle lifted from the space–time tangent bundle $T(X)$ restricted to Σ , $T(X)|_{\Sigma} = T\Sigma \oplus N\Sigma$, where $N\Sigma$ is the normal bundle to Σ in X . The GSO projection restricts the fermions to have definite and opposite chiralities in the $p-p$ and $\bar{p}-\bar{p}$ sectors, and also in the $p-\bar{p}$ and $\bar{p}-p$ sectors, with respect to the local space–time Lorentz group $SO(9, 1)$. Upon dimensional reduction from the space–time manifold X to the worldvolume manifold $\Sigma \subset X$, the space–time Lorentz group is broken to the $(p + 1)$ -dimensional local Lorentz group of Σ plus a global R-symmetry group corresponding to the structure group of the normal bundle $N\Sigma$, i.e. $SO(9, 1) \rightarrow SO(p, 1) \times SO(9-p)$. This implies that a section of $T(X)$ in a representation R when restricted to Σ will decompose into sections of $T\Sigma \otimes N\Sigma$ in representations $R_T^a \otimes R_N^a$. In particular, a chiral spinor field on X will decompose into a multiplet of chiral fermion fields transforming under the adjoint representation

$$\rho = (\mathbf{N}^+ \otimes \overline{\mathbf{N}}^+) \oplus (\mathbf{N}^- \otimes \overline{\mathbf{N}}^-) \oplus (\mathbf{N}^+ \otimes \overline{\mathbf{N}}^-) \oplus (\mathbf{N}^- \otimes \overline{\mathbf{N}}^+) \tag{3.3}$$

of the brane–antibrane Chan–Paton structure group $U(N^+) \times U(N^-)$. This is the structure that is required of the superpartners for the lowest components of the superconnection gauge fields (2.6). The Chan–Paton bundles E^{\pm} are combined into the superbundle $E = E^+ \oplus E^-$ and tensored with the appropriate spinor bundle. The above arguments imply that the total \mathbf{Z}_2 -grading is then the canonical tensor product grading, as in (2.28), i.e. the chiral fermion fields on Σ are the sections

$$\psi = \psi_+ + \psi_-, \quad \psi_{\pm} = \begin{pmatrix} \psi_{(\mp)}^+ \\ \psi_{(\pm)}^- \end{pmatrix} \in C^{\infty}(\Sigma, S_E^{\pm}). \tag{3.4}$$

The fluctuations of these chiral fermion fields lead to quantum anomalies in the brane–antibrane worldvolume effective field theory. There are several ways to argue that these are the only anomalies produced. For instance, one may argue as we have above from the basic properties of the GSO projection, or alternatively by examining the massless spectrum of the intersection of two brane–antibrane systems as we will briefly describe later on. One subtlety concerns the fact that the brane–antibrane system is actually unstable. The fact that one is not sitting in the true vacuum of the theory in making these arguments makes them a little suspect. However, it is believed that supersymmetry is not a necessary requirement in identifying these chiral fermionic zero modes, i.e. even for non-supersymmetric brane configurations it is still possible to correctly capture the massless fermionic content as above [16,20]. In any case, we assume that this is indeed the appropriate ground state that remains after tachyon condensation on the brane–antibrane system. This assumption may

be motivated by the fact that it will lead to the appropriate massless spectrum of induced lower-dimensional D-brane charges that remain after the tachyonic Higgs mechanism.

We may now proceed to use the standard topological index formula [50]. For this, we consider the appropriate spinor bundle $\mathcal{S}(\Sigma) = \mathcal{S}(T\Sigma) \otimes \mathcal{S}(N\Sigma) = \mathcal{S}^+(\Sigma) \oplus \mathcal{S}^-(\Sigma)$, where $\mathcal{S}(T\Sigma)$ and $\mathcal{S}(N\Sigma)$ are the spinor bundles lifted from the tangent and normal bundles to Σ in X , respectively, and we use the standard tensor product grading

$$\mathcal{S}^\pm(\Sigma) = (\mathcal{S}^\pm(T\Sigma) \otimes \mathcal{S}^+(N\Sigma)) \oplus (\mathcal{S}^\mp(T\Sigma) \otimes \mathcal{S}^-(N\Sigma)). \tag{3.5}$$

The open string ground states in the Ramond sector are sections of $\mathcal{S}(T\Sigma)$, while those of the Neveu–Schwarz sector are sections of $\mathcal{S}(N\Sigma)$. We then incorporate the Chan–Paton superbundle by defining $\mathcal{E} = \mathcal{S}(\Sigma) \otimes E_\rho = \mathcal{E}^+ \oplus \mathcal{E}^-$, where ρ is the $U(N^+) \times U(N^-)$ representation (3.3) carried by the fermionic open string zero modes. The appropriate grading is then taken to be

$$\mathcal{E}^\pm = (\mathcal{S}^\pm(\Sigma) \otimes E_\rho^+) \oplus (\mathcal{S}^\mp(\Sigma) \otimes E_\rho^-). \tag{3.6}$$

The Dirac operator \mathcal{D} , constructed as above from the appropriate lifted spinor bundles, defines the two-term complex

$$C^\infty(\Sigma, \mathcal{E}^\pm) \xrightarrow{\mathcal{D}} C^\infty(\Sigma, \mathcal{E}^\mp), \tag{3.7}$$

and the standard index theorem applied to the superbundle $\mathcal{E} \rightarrow \Sigma$ yields

$$\text{index } i\mathcal{D} = (-1)^{(p+1)(p+2)/2} \int_\Sigma \text{ch}^+(\mathcal{E}) \wedge \frac{\text{Td}(T\Sigma \otimes \mathbf{C})}{\chi(T\Sigma)}. \tag{3.8}$$

Here $\text{ch}^+(\mathcal{E})$ denotes the Chern character of the superbundle \mathcal{E} which may be represented by the closed differential form [11]

$$\text{ch}^+(\mathcal{E}) = \text{Tr}^+ \exp \frac{1}{2\pi i} F_{\mathbb{E}}, \tag{3.9}$$

where \mathbb{E} is a superconnection on \mathcal{E} and the supertrace is taken over the \mathbf{Z}_2 -graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, with $\mathcal{H}^\pm = C^\infty(\Sigma, \mathcal{E}^\pm)$.⁴ The grading automorphism (2.19) satisfies $\varepsilon\mathcal{D} = -\mathcal{D}\varepsilon$. The second factor in (3.8) is a standard characteristic class which depends only on the topology of the brane worldvolume manifold. For any oriented real vector bundle $V \rightarrow \Sigma$, $\chi(V)$ denotes the cohomological Euler class of V , i.e. the restriction of the Thom class $\Phi(V)$ to the zero section $H^{p+1}(V, \mathbf{Z}) \cong H^{p+1}(\Sigma, \mathbf{Z})$, while $\text{Td}(V \otimes \mathbf{C})$ is the Todd class of the complexification of V . The latter class may be related to Atiyah–Hirzebruch class $\hat{A}(V)$ by

$$\hat{A}(V) = e^{d(V)/2} \wedge \sqrt{\text{Td}(V \otimes \mathbf{C})}, \tag{3.10}$$

where the degree 2 integral characteristic class $d(V)$ determines a spin^c -structure on V and may be defined as follows. The group homomorphism

$$\text{spin}^c(p+1) = \frac{(\text{spin}(p+1) \times U(1))}{\mathbf{Z}_2} \rightarrow U(1) \tag{3.11}$$

⁴ More precisely, \mathcal{H} is the L^2 -norm completion of the space of smooth sections on $\Sigma \rightarrow \mathcal{E}$ with inner product defined with respect to the Riemannian volume form $d \text{vol}(\Sigma)$ of Σ .

induces a map $H^1(\Sigma, \text{spin}^c(p+1)) \rightarrow H^1(\Sigma, U(1))$ on cohomology, from which we may associate a complex line bundle $L_V \rightarrow \Sigma$ to the vector bundle V . Then $d(V)$ is defined as the first Chern class $c_1(L_V)$ of this line bundle, and its mod 2 reduction yields the second Stiefel–Whitney class $w_2(V)$.

The index formula (3.8) can be expanded by using the fact that the Chern character respects the semi-ring structure as a map $V \mapsto \text{ch}^+(V)$ from bundles to cohomology classes, i.e. $\text{ch}^+(V \oplus W) = \text{ch}^+(V) + \text{ch}^+(W)$ and $\text{ch}^+(V \otimes W) = \text{ch}^+(V) \wedge \text{ch}^+(W)$ so that

$$\text{ch}^+(\mathcal{E}) = \text{ch}_\rho^+(E) \wedge \text{ch}^+(\mathcal{S}(T\Sigma)) \wedge \text{ch}^+(\mathcal{S}(N\Sigma)), \tag{3.12}$$

where ch_ρ^+ denotes the Chern character as in (3.9) but with the supertrace taken in the representation (3.3) of the Chan–Paton structure group. Furthermore, if $V \rightarrow \Sigma$ is any oriented real spin^c -bundle and $\mathcal{S}(V) = \mathcal{S}^+(V) \oplus \mathcal{S}^-(V)$ the spinor bundle lifted from V , then we have the identity

$$\text{ch}^+(\mathcal{S}(V)) = e^{d(V)} \wedge \frac{\chi(V)}{\hat{A}(V)}. \tag{3.13}$$

Combining the above relations, we may write (3.8) finally as

$$\text{index } i\mathcal{D} = \int_\Sigma \text{ch}^+(E) \wedge \text{ch}^+(\bar{E}) \wedge e^{d(N\Sigma)} \wedge \frac{\hat{A}(T\Sigma)}{\hat{A}(N\Sigma)} \wedge \chi(N\Sigma), \tag{3.14}$$

where \bar{E} is the complex conjugate of the superbundle E . The Chern characters in (3.14) may be represented in terms of supertraces in the fundamental representation of $U(N^+) \times U(N^-)$ and the curvature $F_{\mathbb{A}}$ of a superconnection on E [11], as described in the previous section. We have used the properties $\text{ch}_{\rho_1 \otimes \rho_2}^+(V) = \text{ch}_{\rho_1}^+(V) \wedge \text{ch}_{\rho_2}^+(V)$ and $\text{ch}_\rho^+(V) = \text{ch}_\rho^+(\bar{V})$ for unitary gauge bundles $V \rightarrow \Sigma$. Note that the class $d(N\Sigma)$ also determines the spin^c -structure on Σ itself [9,23].

The perturbative chiral gauge anomaly is related to the index (3.14) in the usual way [51,52] by the descent formula $A = 2\pi i(\text{index } i\mathcal{D})^{(1)}$ where, for any invariant polynomial I of the Chan–Paton gauge bundle E , $I^{(1)}$ denotes its Wess–Zumino descendent which is constructed as in Section 2.4. Namely, we decompose $I = I_0 + dI^{(0)}$ locally, where I_0 is the constant part of I and $I^{(0)}$ its secondary characteristic. Then $I^{(1)}$ is defined via the gauge variation $\delta_{\text{BRST}} I^{(0)} = dI^{(1)}$. In fact, the derivation given above can be applied straightforwardly to compute the anomaly on an ordinary D-brane system, by noting that any vector bundle $V \rightarrow \Sigma$ can be trivially \mathbb{Z}_2 -graded by setting $V^+ = V$ and $V^- = \Sigma \times \{0\}$. This defines an even grading with grading automorphism $\varepsilon = 1$. The Chern characters (3.9) then coincide with the usual (ungraded) ones. As is the usual case, these quantum anomalies should cancel the classical anomalies which arise due to the magnetic RR interactions of D-branes. This standard argument [8,19–21] is of a cohomological nature and can be straightforwardly adapted to brane–antibrane systems [16]. We will therefore be brief.

Given a closed brane–antibrane worldvolume Σ , we postulate a coupling to the space–time Ramond–Ramond p -form fields $C_{(p)}$ of the form (3.2), where the coupling \mathcal{Y} is determined by demanding that the classical and quantum anomalies cancel each other. We integrate

(3.2) by parts and rewrite it in terms of the constant part \mathcal{Y}_0 and the descendent $\mathcal{Y}^{(0)}$ of \mathcal{Y} , i.e. $\mathcal{Y} = \mathcal{Y}_0 + d\mathcal{Y}^{(0)}$ with $\delta_{\text{BRST}}\mathcal{Y}^{(0)} = d\mathcal{Y}^{(1)}$. The zero mode \mathcal{Y}_0 may be set to unity by suitably normalizing the charge μ . With $G = d\mathcal{C}$, the total RR field strength, we then have

$$Z = -\frac{\mu}{2} \int_X \delta(\Sigma) \wedge (\mathcal{C} - (-1)^\sigma G \wedge \mathcal{Y}^{(0)}), \tag{3.15}$$

where $\sigma = 1$ (resp. $\sigma = 0$) for Type IIA (resp. Type IIB) D-branes, and $\delta(\Sigma)$ is the de Rham current which is a delta-function supported representative of the Poincaré dual cohomology class to the embedding $\Sigma \hookrightarrow \phi X$. Globally, $\delta(\Sigma)$ is a section of the normal bundle $N\Sigma$ with compact support. It may be represented locally on Σ by taking the zero section of $N\Sigma$. On the other hand, in cohomology $\delta(\Sigma)$ may be identified with the Thom class $\Phi(N\Sigma)$ of the normal bundle whose zero section is the Euler class $\chi(N\Sigma)$. It follows that the de Rham current possesses the global property [20]

$$\delta(\Sigma) \wedge \delta(\Sigma) = \delta(\Sigma) \wedge \chi(N\Sigma). \tag{3.16}$$

The coupling (3.15) modifies the RR equations of motion and Bianchi identity. In particular, G is no longer a closed form, because

$$dG = -\mu\delta(\Sigma) \wedge \bar{\mathcal{Y}}, \tag{3.17}$$

where $\bar{\mathcal{Y}}$ is obtained from \mathcal{Y} by complex conjugation of the corresponding Chan–Paton gauge group representation. The minimal expression for the field strength G is then

$$G = d\mathcal{C} - (-1)^\sigma \mu\delta(\Sigma) \wedge \bar{\mathcal{Y}}^{(0)} \tag{3.18}$$

with $\bar{\mathcal{Y}}^{(0)}$ the secondary characteristic of $\bar{\mathcal{Y}}$. By demanding that G be gauge-invariant, it follows that the potential \mathcal{C} must acquire an anomalous gauge transformation in order to compensate the gauge variation of the second term in (3.18),

$$\delta_{\text{BRST}}\mathcal{C} = \mu\delta(\Sigma) \wedge \bar{\mathcal{Y}}^{(1)}, \tag{3.19}$$

where $\bar{\mathcal{Y}}^{(1)}$ is the Wess–Zumino descendent of $\bar{\mathcal{Y}}$. Thus, under a gauge transformation δ_{BRST} , one finds that the RR couplings (3.15) yield a gauge anomaly given by

$$\delta_{\text{BRST}}Z = -\frac{\mu^2}{2} \int_X \delta(\Sigma) \wedge \delta(\Sigma) \wedge (\mathcal{Y} \wedge \bar{\mathcal{Y}})^{(1)}. \tag{3.20}$$

By using (3.16), we find that the magnetic RR coupling on Σ is anomalous.

The corresponding classical anomaly inflow is given by $A = 2\pi i \int_\Sigma I^{(1)}$, where

$$I = -\frac{\mu^2}{4\pi} \mathcal{Y} \wedge \bar{\mathcal{Y}} \wedge \chi(N\Sigma). \tag{3.21}$$

The anomalous form (3.21) is of the same type as the integrand of (3.14), and it implies that the anomalous RR coupling on the brane–antibrane system is given by

$$\mathcal{Y} = \text{ch}^+(E) \wedge e^{d(N\Sigma)/2} \wedge \sqrt{\frac{\hat{A}(T\Sigma)}{\hat{A}(N\Sigma)}}. \tag{3.22}$$

The classical and quantum anomalies will thereby cancel provided that $\mu = \sqrt{2\pi}$, which is the correct quantum of RR charge (in string units). Furthermore, these choices cancel the anomalies which arise from the magnetic RR interactions between two different brane–antibrane systems, i.e. on the I-branes $\Sigma_{12} = \Sigma_1 \cap \Sigma_2$ [8,19–21]. Then, in addition to the chiral massless fermions required by open string quantization and the GSO projection, there are massless spinors which arise from the open string sectors which start in Σ_1 and end in Σ_2 . Because of the previously stated properties of the GSO projection, the only fermionic zero modes which can arise in this way come from the open string sectors which begin on a brane (resp. antibrane) and end on another brane (resp. antibrane). Thus the index theoretical calculation on an I-brane Σ_{12} will go through in exactly the same way as before, and produce a quantum anomaly which cancels the classical I-brane anomaly inflow as above.

Although the derivation above formally captures the dependence of the RR couplings on the superconnection gauge fields, it should be noted that while they do naturally arise from the pertinent index theorem, the anomaly inflow arguments are not sufficient to determine the tachyon terms in the action.⁵ This follows from the fact that inflow determines the couplings only up to gauge-invariant terms. One can change representative of the cohomology class (3.22) to $\mathcal{Y} + d\mathcal{V}$, where \mathcal{V} is any local, gauge-invariant polynomial. For closed Ramond–Ramond potentials \mathcal{C} , such extra terms do not contribute to the charge. Since the tachyon-dependent terms in the superconnection Chern character (3.9) are precisely of this gauge-invariant form, the tachyon-dependent parts of the RR couplings can be cancelled in this way. Of course, this immediately follows from the fact, proven in Section 2.4, that the cohomology classes generated by $\text{ch}^+(E)$ are independent of the tachyon field T . Put differently, the worldvolume fermionic zero modes are not charged with respect to the tachyon background. Nevertheless, since the usual anomaly analysis, based on the index theorem, applied to the worldvolume gauge fields involves the Chern character of a superbundle, we shall choose the most general representative which involves a non-zero tachyon field T . As we will see, this formula leads to the correct results for the tachyonic couplings on non-BPS systems of branes and, moreover, determines the appropriate relationship between D-brane charges and K-homology.

3.2. Complete Chern–Simons action

Before writing down the final version of the anomalous coupling on a brane–antibrane system, there are some aspects of the above derivation that we should first discuss. We shall work in the static gauge of the worldvolume diffeomorphism group which may be defined as follows. We split the local coordinates of X into longitudinal and transverse coordinates with respect to Σ , $x^a = (x^\mu, x^i)$, and use space–time diffeomorphism invariance to fix Σ at the coordinates $x^i = 0$ for $i = p + 1, \dots, 9$. Then we use worldvolume diffeomorphism invariance to identify the longitudinal coordinates with those of Σ , $x^\mu = \xi^\mu$ for $\mu = 0, 1, \dots, p$. For multiple branes and antibranes, we should identify the transverse coordinates to the worldvolume Σ with matrices in the Lie algebra of the brane–antibrane

⁵ The author is grateful to F. Larsen for pointing out this to him.

gauge group $U(N^+) \times U(N^-)$, $x^i = \mathcal{X}^i$, where

$$\mathcal{X}^i = \begin{pmatrix} \phi_+^i & 0 \\ 0 & \phi_-^i \end{pmatrix}. \tag{3.23}$$

The $N^+ \times N^+$ (resp. $N^- \times N^-$) Hermitian matrices ϕ_+^i (resp. ϕ_-^i) describe the $9 - p$ transverse degrees of freedom of the branes (resp. antibranes). They transform in the adjoint representation of $U(N^+)$ (resp. $U(N^-)$) and correspond to the fields of the R-symmetry group $SO(9 - p)$ of the dimensionally reduced Yang–Mills gauge theory to the brane–antibrane worldvolume. From a global perspective, we may use the Riemannian structure on the space–time manifold X to identify the normal bundle $N\Sigma$ with a tubular neighbourhood of the worldvolume Σ in X . Then the transverse degrees of freedom of the brane–antibrane system wrapping Σ which are described by sections of $N\Sigma$ are augmented to sections of the $U(N^+) \times U(N^-)$ bundle $N\Sigma \otimes (\text{End}(E^+) \oplus \text{End}(E^-))$. Note that the total brane–antibrane scalar fields (3.23) are block-diagonal because the GSO projection eliminates the scalar degrees of freedom in the $p - \bar{p}$ open string sectors. In these sectors there are of course the remnant tachyon fields, but these are objects which live in the worldvolume theory itself and are not attributed to the transverse degrees of freedom of the branes and antibranes. We shall see below how to include the modifications due to the tachyonic degrees of freedom.

The standard definition of pullbacks should then be altered so as to replace all transverse coordinates with the matrices \mathcal{X}^i and all worldvolume derivatives with covariant ones [26,27]. In order to obtain a covariant expression, we must also account for the possible non-trivial normal bundle topology and the fact that the transverse scalar fields are really sections of $N\Sigma$. Let α_i^m , $m = p + 1, \dots, 9$, span a frame in $N\Sigma$, and introduce the connection $\Theta_{\mu n}^m = \alpha_j^m \partial_\mu \alpha_n^j$ on the normal bundle to the brane–antibrane worldvolume. Then, on the Ramond–Ramond fields, the definition of the pullback ϕ^* induced by the embedding $\Sigma \hookrightarrow \phi X$ is taken to be

$$\begin{aligned} &(\phi^* C_{(p)})_{\mu_1 \dots \mu_p} \\ &= C_{(p)\mu_1 \dots \mu_p} + \sum_{k=1}^p C_{(p)i_1 \dots i_k \{\mu_1 \dots \mu_{p-k}\}} \nabla_{\mu_{p-k+1}}^N \mathcal{X}^{m_1} \alpha_{m_1}^{i_1} \dots \nabla_{\mu_p}^N \mathcal{X}^{m_k} \alpha_{m_k}^{i_k}, \end{aligned} \tag{3.24}$$

where

$$\nabla_\mu^N \mathcal{X}^m \alpha_m^i = (D_\mu \mathcal{X}^m + \Theta_{\mu n}^m \mathcal{X}^n) \alpha_m^i, \tag{3.25}$$

and D_μ is the covariant derivative defined as in (2.16). This produces a non-trivial interaction between the Ramond–Ramond fields and the non-abelian transverse excitations of the branes and antibranes. In the case of multiple D-branes alone, there are in addition multipole terms and other commutator terms which couple to the background supergravity fields [26]. These terms are required by T-duality and in order to match results from Matrix theory. In the present case, we will impose such requirements, in addition to the pullback definition (3.24), for the sake of matching with the recent observations concerning D-brane effective actions. The matrix structure of the transverse coordinates for multiple branes and antibranes will become important later on and will lead to a D-brane action which is explicitly T-duality-invariant.

First of all, the background space–time fields restricted to the worldvolume Σ are formally regarded as functions of the transverse coordinates, under the identification $x^i = \mathcal{X}^i$. This is achieved by using the formal Taylor series expansions of the fields in the transverse coordinates and it defines couplings of their multipole moments to the adjoint scalar fields \mathcal{X}^i . For instance, for the RR tensor potentials we write

$$C_{(p)a_1 \dots a_p} = C_{(p)a_1 \dots a_p}(\xi^\mu, 0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_n}} C_{(p)a_1 \dots a_p}(\xi^\mu, x^i) \Big|_{x^i=0} \mathcal{X}^{i_1} \dots \mathcal{X}^{i_n}. \tag{3.26}$$

The couplings in (3.26) may be obtained by application of the operator $\exp \mathcal{J}_{\mathcal{X}} d_{\perp}|_{x^{\perp}=0} = \exp \mathcal{X}^i (\partial/\partial x^i)|_{x^i=0}$ to the field $C_{(p)}$, where $\mathcal{J}_{\mathcal{X}}$ is the interior multiplication operator with respect to \mathcal{X}^i of degree -1 and d_{\perp} the exterior derivative on the normal bundle $N\Sigma$. Of course, in the case $N^+ = N^- = 1$, the scalar fields ϕ_+^i, ϕ_-^i coincide with the transverse coordinates x^i themselves and all higher partial wave couplings in (3.26) disappear. In addition, we need to insert a coupling of the background fields to commutators of the scalar fields (3.23). This is again achieved via action of the interior multiplication operator as $e^{i(\mathcal{J}_{\mathcal{X}})^2} C_{(p)}$. Using antisymmetry of the components of the differential form $C_{(p)}$, the n th term (with $2n \leq p$) in the expansion of this object is given by

$$[(\mathcal{J}_{\mathcal{X}})^{2n} C_{(p)}]_{a_1 \dots a_{p-2n}} = \frac{1}{2^n \binom{p}{2n}} C_{(p)i_1 \dots i_{2n} a_1 \dots a_{p-2n}} [\mathcal{X}^{i_1}, \mathcal{X}^{i_2}] \dots [\mathcal{X}^{i_{2n-1}}, \mathcal{X}^{i_{2n}}]. \tag{3.27}$$

In addition, T-duality invariance requires the background fields to couple to the tachyon field, because such couplings are induced by T-duality transformations of the one-form parts $\mathcal{F}_{(1)}$ of the superconnection field strengths in (2.13) [6,32]. The appropriate modification comes from replacing the operator $(\mathcal{J}_{\mathcal{X}})^2$ by

$$\mathcal{J}_{\mathcal{X}}(T)^2 = (\mathcal{J}_{\mathcal{X}})^2 + i \left[\mathcal{J}_{\mathcal{X}}, \begin{pmatrix} 0 & T^{\dagger} \\ T & 0 \end{pmatrix} \right]. \tag{3.28}$$

Having described the appropriate physical alterations of the anomalous couplings which must be made for multiple branes and antibranes, we now turn to a discussion of the factor $e^{d(N\Sigma)/2}$ in (3.22), which accounts for the spin^c -structure on Σ and can induce charge shifts of degree 2 on the brane–antibrane worldvolume Σ . If Σ is a connected almost complex manifold, then the class $d(N\Sigma) \in H^2(\Sigma, \mathbf{Z})$ can be represented by the first Chern class of the normal bundle as $d(N\Sigma) = -c_1(N\Sigma)$. If the worldvolume Σ were not a spin^c -manifold, then one would have to incorporate a topologically non-trivial Neveu–Schwarz two-form field B into the string background in order to cancel certain worldsheet anomalies [9,22,23]. The result of this cancellation is that it trivializes a certain line bundle over the loop space of the brane worldvolume which is defined by two-cycle holonomies of B . In the purely bosonic case this would imply that the B -field restricted

to Σ is necessarily topologically trivial [23]. For the full superstring theory, this is not the case, but the topological type of B restricted to Σ is uniquely determined by the bundle trivialization property. A B -field on Σ is classified topologically by its characteristic class $\phi^*[H] \in H^3(\Sigma, \mathbf{Z})$ which as a differential form is represented by the pullback of the field strength $H = dB$. B being topologically trivial means that $\phi^*[H]$ vanishes as an integral cohomology class, and not only in real cohomology. The bundle trivialization just mentioned is equivalent to the condition $\phi^*[H] = W_3(\Sigma)$, where $W_3(\Sigma) \in H^3(\Sigma, \mathbf{Z})$ is the Dixmier–Douady-invariant which may be defined as the image of the second Stiefel–Whitney class $w_2(\Sigma) \in H^2(\Sigma, \mathbf{Z}_2)$ under the appropriate connecting homomorphism (the Bockstein map) in cohomology [9,22,23]. The cancellation of global worldvolume anomalies in the previous section required that Σ admit a spin^c -structure, which is equivalent to $W_3(\Sigma) = 0$. Thus the restriction of B to Σ is topologically trivial. Furthermore, in that case, in order to cancel the worldsheet anomalies requires that the brane worldvolume gauge fields define spin^c -connections, rather than single-valued ones. But this is precisely what we assumed in Section 2 when we used the superconnection gauge field \mathcal{A} to define a Clifford superconnection \mathbb{S} , and ultimately the appropriate Dirac operator \mathbb{D} which allowed us to compute the anomalous couplings to the RR fields. These remarks illustrate the consistency of the present analysis thus far. It would be interesting to extend the analysis to topologically non-trivial B -fields and hence worldvolumes Σ which do not admit spin^c -structures.

For the purposes of analysing the anomalies in the brane–antibrane worldvolume field theories, we should therefore incorporate a topologically trivial Neveu–Schwarz two-form field. This is included via two-cycle holonomies of B . The final object we need to take special care of is the topological normal bundle correction term in (3.22). This can be properly incorporated by covariantizing the couplings that we have described above [27]. Defining $\mathcal{C}' = \mathcal{C} \wedge e^{-\phi^* B/2\pi i}$, taking into account the non-abelian dynamics of D-branes amounts (for topologically trivial NS–NS B -field) to replacing in action (3.2) the exterior product $\mathcal{C}' \wedge \mathcal{Y}$ by Clifford multiplication defined through the symbol map

$$\sigma_{\mathcal{C}'}(\mathcal{Y}) = \mathcal{C}' \wedge \mathcal{Y} - \mathcal{J}_{\mathcal{C}'} \mathcal{Y}, \tag{3.29}$$

when the couplings are expressed in terms of bulk quantities as in (1.2). This overall modification by the spin geometry of X fits very nicely into the present formalism. However, we will not write the Clifford multiplication explicitly and only assume its presence implicitly when we write exterior products with \mathcal{C}' .

We have thus found that the anomalous, Chern–Simons couplings on a brane–antibrane system wrapping a worldvolume Σ of dimension $p + 1$ are given by

$$\begin{aligned} Z = & -\frac{\sqrt{2\pi}}{2} \int_{\Sigma} \text{Tr}(-1)^F \phi^* \\ & \times \left(\exp \left\{ i(\mathcal{J}_{\mathcal{X}})^2 - \left[\mathcal{J}_{\mathcal{X}}, \begin{pmatrix} 0 & T^\dagger \\ T & 0 \end{pmatrix} \right] \right\} e^{\mathcal{J}_{\mathcal{X}} d_\perp} \Big|_{x^\perp=0} \sum_{p=0}^{4+\sigma} C_{(2p+1-\sigma)\wedge} e^{-B/2\pi i} \right) \\ & \wedge \exp \frac{1}{2\pi i} F_{\mathbb{A}} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \wedge e^{d(N\Sigma)/2}. \end{aligned} \tag{3.30}$$

The trace in (3.30) is taken in the fundamental representation of the $U(N^+) \times U(N^-)$ gauge group, with $(-1)^F$ the grading automorphism of the brane–antibrane pairs and $F_{\mathbb{A}}$ the field strength (2.20) of the superconnection which depends on the worldvolume gauge fields and the brane–antibrane tachyon field (here we take $\mathcal{G} = \mathbb{1}$ in (2.38)). The matrix products in the argument of the trace in (3.30) must be given an appropriate ordering prescription, which we take to be the symmetrized trace defined by (2.53) [26]. This trace symmetrization will also be implicitly assumed in the following. The third term in (3.30) gives the appropriate gravitational couplings of the fields, with R_T (resp. R_N) the Riemann curvature two-form of the tangent (resp. normal) bundle, and

$$\hat{A}(R) = \prod_{a \geq 1} \frac{r_a}{\sinh r_a}, \tag{3.31}$$

where $4\pi r_a$ are the skew-eigenvalues of R_{ab} . Action (3.30) agrees with the form of the brane–antibrane coupling originally proposed in [14]. Notice, however, that the formula (3.30) is ambiguous with respect to on-shell terms such as the equation of motion for the tachyon field. The superconnection formalism employed above gives a unique off-shell prescription for the RR charges, alternatively to the boundary string field theory prescription used in [5]. The physical implications of this will be that the index theoretical calculations automatically lead to the relationships between D-brane charges and K-theory, as will be extensively described in Section 6.

3.3. Tachyon condensation

There is another natural-invariant action that can be constructed using the superconnection formalism. For this, we consider the natural inner product density (\cdot, \cdot) on the algebra $\Omega(\Sigma, \text{End } E)$ of sections of the endomorphism bundle defined by

$$(A, B) = \text{Tr}(B^\dagger \wedge *A), \tag{3.32}$$

where $*$ denotes the Hodge star-operator on Σ . This is the inner product density that is canonically inherited from $\Omega(\Sigma)$. If $\|\cdot\|$ denotes the corresponding norm density, then we can write down a Euclidean action in the form

$$Z_{\text{kin}} = \frac{1}{2} \int_{\Sigma} \|F_{\mathbb{A}} - \mathcal{G}\|^2 = \frac{1}{2} \int_{\Sigma} \left(\sum_{k \geq 1} \|\mathcal{F}_{(k)}\|^2 + \|\mathcal{F}_{(0)} - \mathcal{G}\|^2 \right), \tag{3.33}$$

where

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}^+ & 0 \\ 0 & \mathcal{G}^- \end{pmatrix} \tag{3.34}$$

is a constant abelian flux with $(\mathcal{G}^\pm)^\dagger = \mathcal{G}^\pm$. Expanding (3.33) out using (2.12)–(2.14) then leads to

$$Z_{\text{kin}} = \int_{\Sigma} \text{dvol}(\Sigma) \text{Tr} \left[\frac{1}{2} (F_{\mu\nu}^+)^2 + \frac{1}{2} (F_{\mu\nu}^-)^2 + |D_\mu T|^2 + \frac{1}{2} (T^\dagger T - \mathcal{G}^+)^2 + \frac{1}{2} (T T^\dagger - \mathcal{G}^-)^2 + \dots \right]. \tag{3.35}$$

This is the general form of the brane–antibrane worldvolume action anticipated from two-loop order, on-shell string theory scattering amplitudes [53]. Therefore, we see that the superconnection formalism also gives a compact way of representing the kinetic terms in the low-energy effective field theory. Terms involving the worldvolume scalar fields (3.23) may be incorporated in a manner analogous to that described in the previous section [26]. Higher-order corrections to (3.35) presumably come from a Born–Infeld expansion in powers $(F_A)^n$ of the superconnection curvature [15,34,35].

In particular, from the second line of (3.35) we obtain an explicit expression for the tachyon potential. The minima of action (3.33) determine the tachyon condensates T_c which are given by the equation

$$F_A = \mathcal{G}. \tag{3.36}$$

This condition requires, among other things, covariantly flat gauge field configurations on the branes and antibranes, and a covariantly constant tachyon field. There are two special cases whereby the variational equation (3.36) can be solved with ease. If $\mathcal{G} = 0$, there is a unique solution $T_c = 0$ giving a $U(N^+) \times U(N^-)$ -invariant vacuum. In this case there is no symmetry breaking. If $\mathcal{G} = m^2 \mathbb{1}$ with $m^2 > 0$ and $N^+ = N^- = N$, then $T_c^\dagger T_c = T_c T_c^\dagger = m^2 \mathbb{1}$ and T_c establishes an isomorphism between the fibre spaces E_x^+ and E_x^- of the Chan–Paton superbundle $E \rightarrow \Sigma$. Conversely, any such isomorphism yields a tachyon condensate T_c . This configuration breaks the $U(N) \times U(N)$ gauge symmetry group down to its diagonal subgroup $U(N)_{\text{diag}}$. The constant operator \mathcal{G} is a constant abelian flux on the brane–antibrane worldvolume, and the tachyon field is asymptotically a bundle isomorphism between the branes and antibranes where it reaches its vacuum expectation value. Thus we recover the standard requirements for tachyon condensation in brane–antibrane systems [1,2,9,10,54]. Here we have derived them from a purely geometric formalism, which also allows for more general symmetry breaking patterns.

The mechanism for symmetry breaking here is even more elementary, because it in fact originates from the generalized Dirac operator (2.37)–(2.39). To see this, we use the Lichnerowicz formula for the ordinary Dirac operators (2.39) to compute

$$\begin{aligned} \mathbb{D}^2 = & \left(\Delta_\Sigma + \frac{1}{4} r_\Sigma \right) \mathbb{1}_{N^+|N^-} \\ & + \begin{pmatrix} Q(F^+ + T^\dagger T) & \mathbb{D}_A^+(\mathcal{G}_s \otimes T^\dagger) + (\mathcal{G}_s \otimes T^\dagger) \mathbb{D}_A^- \\ \mathbb{D}_A^-(\mathcal{G}_s \otimes T) + (\mathcal{G}_s \otimes T) \mathbb{D}_A^+ & Q(F^- + TT^\dagger) \end{pmatrix}, \end{aligned} \tag{3.37}$$

where Δ_Σ is the Laplace–Beltrami operator, r_Σ the scalar curvature of the worldvolume Σ and $Q = c \circ \sigma^{-1}$ the quantization map. From (3.37), it follows that the Lagrangian of (3.33) may be computed from the symbol map as

$$\|F_A\|^2 = \text{Tr}[\sigma \circ c^{-1}(\mathbb{D}^2)]^2. \tag{3.38}$$

Therefore, both the topological and the Born–Infeld type action on the brane–antibrane system can be derived from the Dirac operator \mathbb{D} . This is not surprising, since as we mentioned in Section 2.3 the Dirac operator carries the same amount of information as its corresponding superconnection [13]. This property is even more apparent if we supersymmetrize action

(3.33) by adding a fermion coupling $\bar{\Psi} \mathbb{D} \Psi$, with Ψ the spinor fields (3.4). Then the fermion masses are induced by the Dirac operator and correspond to the tachyonic expectation values of the brane–antibrane pairs. The mass matrix $\mathcal{G}_s \otimes T_c$ of the fermion fields originate from the quantum field theory of the $p-\bar{p}$ open string ground states which are given by the operator (2.38) [55]. Therefore, all the standard properties of tachyon condensation come from a spectral action involving the relevant generalized Dirac operator. These facts will be instrumental in the K-theory interpretation that we shall give in Section 6. Notice, however, that (3.35) is *not* the full form of the kinetic part of the worldvolume action. In particular, it only agrees with the results of [5] up to terms involving off-shell and also field-redefinition ambiguities. The assumption that the kinetic terms can be written in terms of superconnection (or, equivalently, generalized Dirac operator) quantities alone implicitly imposes an on-shell requirement.

Having established that the unstable brane–antibrane system will decay via a Higgs mechanism, let us now examine the corresponding reduction of the Chern–Simons action (3.30). First of all, we note that if the tachyon field is absent, $T \equiv 0$, then action (3.30) is a sum $Z = Z_+ + Z_-$, where

$$\begin{aligned}
 Z_{\pm} = & \mp \frac{\sqrt{2\pi}}{2} \int_{\Sigma} \text{Tr}_{\pm} \phi_{\pm}^* \left(e^{i(\mathcal{J}_{\phi_{\pm}})^2} e^{\mathcal{J}_{\phi_{\pm}} d_{\perp}} \Big|_{x^{\perp}=0} \sum_{p=0}^{4+\sigma} C_{(2p+1-\sigma)} \wedge e^{-B/2\pi i} \right) \\
 & \wedge \exp \frac{1}{2\pi i} F^{\pm} \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \wedge e^{d(N\Sigma)/2}.
 \end{aligned} \tag{3.39}$$

The \pm indices label the contributions from the branes and antibranes, respectively, so that Tr_{\pm} denotes the (symmetrized) trace in the fundamental representation of $U(N^{\pm})$. Thus when the branes are well separated from the antibranes (so that there are no massless open string $p-\bar{p}$ modes), the total Ramond–Ramond charge is given as the sum of RR charges on the branes and antibranes. This includes the extra multipole couplings for non-abelian systems as is required by T-duality [26]. Similarly, in this case action (3.35) decomposes into a sum of Yang–Mills actions for the field strengths F^{\pm} on the branes and antibranes. An interesting feature to examine in this context is the critical value of the tachyon field at which the open string $p-\bar{p}$ modes become relevant again and $Z \neq Z_+ + Z_-$ [39]. Within the present framework this is a difficult question to answer, however, because one would need to use distinguishable D-branes with distinct worldvolume manifolds for the branes and antibranes [53].

Now let us reinstate the tachyonic coupling and see how to realize a $(p - 2k)$ -brane in the worldvolume Σ of the $p-\bar{p}$ pairs [9,10]. For this, we assume that all length scales of the problem are much larger than the string scale. We set $N^+ = N^-$ and let $\tilde{\Sigma}$ be a spin^c -submanifold of co-dimension $2k$ in Σ . Then the normal bundle $N(\tilde{\Sigma}, \Sigma)$ to $\tilde{\Sigma}$ in Σ has structure group $\text{SO}(2k)$. Let $\tilde{\Sigma}'$ be a tubular neighbourhood of $\tilde{\Sigma}$ in Σ . Since $\tilde{\Sigma}$ and Σ only have spin^c -structures defined on them, the spinor bundles $\mathcal{S}^{\pm}(N(\tilde{\Sigma}, \Sigma))$ cannot in general be constructed globally. Let $L_N \rightarrow \tilde{\Sigma}$ be the complex line bundle corresponding to the integral cohomology class $d(N(\tilde{\Sigma}, \Sigma))$, i.e. $c_1(L_N) = d(N(\tilde{\Sigma}, \Sigma))$. Then, generally, the square root $L_N^{1/2}$ (with $L_N^{1/2} \otimes L_N^{1/2} = L_N$) also cannot be constructed globally. When

there is two-torsion in the cohomology group $H^2(\tilde{\Sigma}, \mathbf{Z})$, there are different square roots of L_N and hence more than one spin^c-structure for a given class $d(N(\tilde{\Sigma}, \Sigma))$. However, the twisted spinor bundles

$$\mathcal{S}_{L_N^{1/2}}^\pm = L_N^{1/2} \otimes \mathcal{S}^\pm[N(\tilde{\Sigma}, \Sigma)] \tag{3.40}$$

do exist as vector bundles over $\tilde{\Sigma}'$. This is the precise meaning of the existence of a spin^c-structure for $\tilde{\Sigma}$, and also of the vanishing of the global worldsheet anomaly for topologically trivial B -field [23]. We recall once again from Section 2 that the natural geometrical objects on the bundle (3.40) are Clifford superconnections, or equivalently generalized Dirac operators.

Let $\mathcal{L} \rightarrow \tilde{\Sigma}$ be a given complex line bundle. We extend \mathcal{L} over all of Σ , if necessary by using Swan’s theorem to choose a bundle $I \rightarrow \tilde{\Sigma}$ such that $\mathcal{L} \oplus I$ is trivial. Similarly, if necessary, we choose a bundle $\mathcal{I} \rightarrow \tilde{\Sigma}$ such that $\mathcal{S}_{L_N^{1/2}}^- \oplus \mathcal{I}$ is trivial. Then both $\mathcal{L} \oplus I$ and $\mathcal{S}_{L_N^{1/2}}^- \oplus \mathcal{I}$ are extendable as vector bundles to the whole of Σ . For the Chan–Paton superbundle $E = E^+ \oplus E^-$ over Σ , we may then take

$$E^\pm = \mathcal{L} \otimes \mathcal{S}_{L_N^{1/2}}^\pm \oplus I \oplus \mathcal{I}. \tag{3.41}$$

To construct a tachyon field, we consider the generators $\Gamma_i, i = 1, \dots, 2k$, of the complex Clifford algebra $C\ell_{2k}$ of the transverse structure group, which satisfy the Euclidean–Dirac algebra

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}, \tag{3.42}$$

and which may be decomposed as

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_i^\dagger \\ \gamma_i & 0 \end{pmatrix} \tag{3.43}$$

with respect to the chirality \mathbf{Z}_2 -grading of the spinor bundle. They are viewed as elements of the unitary group $U(2^k)$. The tachyon field is then the section of E which is defined locally by Clifford multiplication as

$$T(x) = \mathbb{1}_{\mathcal{L}} \otimes \left(2\pi f(x) \sum_{i=1}^{2k} \gamma_i x^i \right) \oplus \mathbb{1}_{I \oplus \mathcal{I}} \tag{3.44}$$

for $x \in \tilde{\Sigma}'$, where $f(x)$ is a real-valued convergence factor which is constant near $\tilde{\Sigma}$ (where $x^i = 0$) and which behaves as $f_\infty/\|x\|, f_\infty = \text{const.}$, near $\partial\tilde{\Sigma}'$ (where $x^i \rightarrow \infty$). We then pick gauge connections A^\pm on E^\pm which satisfy the finite energy conditions $DT = DT^\dagger = 0$ near $\partial\tilde{\Sigma}'$. These choices are the standard assumptions used for tachyon condensation and the bound-state construction of D-branes [1,2,9,10,54]. Note that by substituting the profile (3.44) into (2.12) and using the Clifford relations (3.42) and (3.43), the zero-form component of the supercurvature may be computed to be

$$\mathcal{F}_{(0)}(x) = (2\pi f(x))^2 \Gamma_i \Gamma_j x^i x^j = (2\pi f(x))^2 \|x\|^2 \mathbb{1}. \tag{3.45}$$

The fact that (3.45) is proportional to the identity matrix in Chan–Paton space is related to the fact that the brane–antibrane pairs with the tachyon configuration (3.44) condense to a single brane of co-dimension $2k$. On $\partial\tilde{\Sigma}'$, where the tachyon field (3.44) assumes its vacuum expectation value, the Higgs mass may thereby be determined explicitly to be $m^2 = (2\pi f_\infty)^2$.

The resulting configuration represents the desired superconnection \mathbb{A} to be used in the topological action (3.30), where we now neglect the non-abelian transverse corrections (and hence generically ruin T-duality invariance). The gravitational couplings may be simplified by using multiplicativity of the characteristic classes, along with the Whitney sum decompositions $T\Sigma|_{\tilde{\Sigma}} = T\tilde{\Sigma} \oplus N(\tilde{\Sigma}, \Sigma)$ and $N\tilde{\Sigma} = N(\tilde{\Sigma}, \Sigma) \oplus N\Sigma$. The Chern character may be simplified by using multiplicativity, the definition (3.41), and (3.45) to compute the superconnection field strength (2.20) with the finite energy conditions. It is then straightforward to arrive at

$$\begin{aligned}
 Z = & -\frac{\sqrt{2\pi}}{2} \int_{\tilde{\Sigma}} \sum_{p=0}^{4+\sigma} \phi^*(C_{(2p+1-\sigma)} \wedge e^{-B/2\pi i}) \wedge \text{ch}(\mathcal{L}) \wedge \sqrt{\frac{\hat{A}(T\tilde{\Sigma})}{\hat{A}(N\tilde{\Sigma})}} \\
 & \wedge e^{d(N\tilde{\Sigma})/2} \int_{\tilde{\Sigma}'} e^{-2\pi f(x)^2 \|x\|^2} \hat{A}[N(\tilde{\Sigma}, \Sigma)] \wedge e^{-c_1(L_N)/2} \\
 & \wedge \left[\text{ch} \left(\mathcal{S}_{L_N^{1/2}}^+ \right) - \text{ch} \left(\mathcal{S}_{L_N^{1/2}}^- \right) \right], \tag{3.46}
 \end{aligned}$$

where in this formula ch is the ordinary (ungraded) Chern character. The second integral in (3.46) can be simplified by noting that the gauge field A^- on the antibranes may be taken to be trivial so that $\text{ch}(\mathcal{S}_{L_N^{1/2}}^-) = 2^{k-1}$, while A^+ may be chosen to ensure that the appropriate degree component of the Chern character $\text{ch}(\mathcal{S}_{L_N^{1/2}}^+)$ is non-vanishing so as to produce a non-zero integral over the transverse directions $\tilde{\Sigma}'$. Because of the assumed properties of the function $f(x)$, this integral always converges, and thereby simply produces a constant density factor in (3.46). This leaves only the first integral of (3.46), which is the standard Chern–Simons action for a single BPS $D(p - 2k)$ -brane wrapping a worldvolume $\tilde{\Sigma}$ and with $U(1)$ Chan–Paton gauge bundle \mathcal{L} in Type II superstring theory (the density factor then yields the appropriate tension). Thus the topological action (3.30) correctly reproduces the charge formula for the D-branes obtained via tachyon condensation from the bound state of higher-dimensional brane–antibrane pairs. In this context, since Clifford superconnections can be thought of as quantizations of ordinary connections [11,13], a BPS D-brane may be regarded as the “classical limit” of a non-BPS brane–antibrane system.

This construction can be generalized by including the non-abelian transverse scalar fields (3.23), which are sections of $N\Sigma \otimes (\text{End}(E^+) \oplus \text{End}(E^-))$, and by replacing the complex line bundle \mathcal{L} in (3.41) by a bundle \mathcal{R} of rank

$$M = \frac{N - \text{ch}_0(I) - \text{ch}_0(\mathcal{I})}{2^{k-1}}, \tag{3.47}$$

where ch_0 is the rank function and $N = N^+ = N^-$. (This of course requires a quantization condition on the ranks of the bundles in (3.41) in multiples of 2^{k-1} .) Under the

stated properties of the spin^c-bundles $\mathcal{S}_{L_N^{1/2}}^\pm$ above, it is possible to choose an appropriate configuration of the scalar fields \mathcal{X} such that the charge formula (3.30) coincides with the non-abelian Chern–Simons action for a system of M BPS $D(p - 2k)$ -branes with $U(M)$ Chan–Paton gauge bundle \mathcal{R} and including the T-duality-invariant modifications from the adjoint sections of $N \tilde{\Sigma} \otimes \text{End } \mathcal{R}$. This generalizes the result (3.46) to multiple branes and realizes the M $D(p - 2k)$ -branes as generalized instanton-like configurations via tachyon condensation. In particular, it is possible to realize the Myers dielectric effect [26] whereby M $D(p - 2k)$ -branes expand into a $D(p - 2k + 2r)$ -brane (with M units of worldvolume instanton-like density) in terms of N $p - \bar{p}$ pairs opening up into $2^{k-1} (p + 2r) - (p + 2r)$ pairs. Some details of this construction can be found in [56]. Alternatively, the non-abelian dielectric couplings can be induced by adding the terms $\gamma_i \phi^i$ to the tachyon profile (3.44) [17].

4. Ramond–Ramond couplings on unstable D-branes

In this section, we will derive the Ramond–Ramond couplings on systems of non-BPS D-branes in Type II superstring theory. We will present two complimentary derivations of these actions. The first one is based on the old geometric approach to Higgs fields through dimensional reduction [57]. In its simplest setting, this technique introduces a single extra flat, translationally invariant dimension. The tachyon field is then regarded as the component of the gauge field along the extra direction. The main drawback of this approach is that the superconnection gauge field should not depend on the auxiliary coordinate so that some of the physical information is lost through the dimensional reduction that is encoded in the modes associated to the extra dimension. This problem is cured by a second derivation of the Chern–Simons actions through a particular reduction of the superconnection couplings of the previous section. While this second approach is geometrically appealing because it puts all of the non-supersymmetric configurations of D-branes in Type II superstring theory into a common mathematical framework, it is the dimensional reduction mechanism that plays the role in processes involving tachyon condensation.

4.1. Dimensional reduction

In this section, we will derive the result for Type IIB D-branes and extend it to the Type IIA case via T-duality. Consider a system of N coincident non-BPS Dp -branes, with p even. The mathematical description of this system is much different than that of the brane–antibrane system, mainly because the low-energy field content is drastically altered. As we will now demonstrate, one way to think about this configuration is as the dimensional reduction of a gauge theory in one higher dimension, rather than as a superconnection gauge theory. This lends a somewhat different interpretation to the tachyon field instability present in these systems.

The low-energy field content on the unstable system of branes consists of a $U(N)$ gauge field A_μ , and a Hermitian tachyon field T which transforms in the adjoint representation $\mathbf{N} \otimes \bar{\mathbf{N}}$ of the Chan–Paton gauge group [28,34,58]. The field A_μ is a connection of an

$U(N)$ gauge bundle $E \rightarrow \Sigma$ over the $(p + 1)$ -dimensional worldvolume Σ of the N unstable Dp -branes. There is also a pair of massless, 16-component fermion fields ψ_1, ψ_2 which live in the adjoint representation of $U(N)$ (coming from the massless Yang–Mills and tachyonic supermultiplets). They are associated with the two possible Chan–Paton factors carried by the open strings on each brane. The crucial issue concerns the chiralities of these spinor fields under the local space–time Lorentz group $SO(9, 1)$. In the static gauge, only an $SO(p, 1) \times SO(9 - p)$ subgroup of $SO(9, 1)$ is realized as a manifest symmetry of the worldvolume field theory. Since p is even, neither $SO(p, 1)$ nor $SO(9 - p)$ has a chiral spinor representation, and the GSO projection cannot determine the $SO(9, 1)$ chirality of the fermion fields. Thus, both a left-handed and a right-handed Majorana–Weyl spinor of $SO(9, 1)$ will transform in the same spinor representation of $SO(p, 1) \times SO(9 - p)$, even though the fermion zero modes from the two Chan–Paton sectors of the open string spectrum on each D-brane have the opposite GSO projection. Therefore, the low-energy worldvolume field theory contains a pair of fermion fields ψ_1, ψ_2 each transforming in, say, the right-handed Majorana–Weyl spinor representation of $SO(9, 1)$.

Let us now consider the dimensional extension of the worldvolume Σ to the $(p + 2)$ -dimensional manifold

$$\hat{\Sigma} = \Sigma \times \mathbf{S}^1, \tag{4.1}$$

and coordinatize the circle \mathbf{S}^1 by $y \in [0, 1]$. The pair (A_μ, T) may then be thought of as the dimensional reduction to Σ of an $U(N)$ gauge field \hat{A}_M on $\hat{\Sigma}$ [29]

$$\hat{A}_M = (A_\mu, T), \quad M = 0, 1, \dots, p, y. \tag{4.2}$$

The field (4.2) may be regarded as a connection of an $U(N)$ gauge bundle $\hat{E}_\rho \rightarrow \hat{\Sigma}$, where $\rho = \mathbf{N} \otimes \bar{\mathbf{N}}$ is the $U(N)$ representation carried by the fermionic open string zero modes. Thus the tachyon field T on a system of non-BPS D-branes may be regarded as a gauge connection of the external space \mathbf{S}^1 , induced by a sort of Kaluza–Klein mechanism from the reduction $\Sigma \times \mathbf{S}^1 \rightarrow \Sigma$. This is to be contrasted with the brane–antibrane system, in which the tachyon field was regarded as a gauge connection of the discrete internal space \mathbf{Z}_2 , arising from a sort of Kaluza–Klein mechanism from the reduction $\Sigma \times \mathbf{Z}_2 \rightarrow \Sigma$.

The pair of spinor fields ψ_1, ψ_2 may be likewise regarded as the dimensional reduction of a 32-component Majorana fermion field on $\hat{\Sigma}$,

$$\hat{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{4.3}$$

Therefore, the perturbative chiral gauge anomaly of the worldvolume field theory on Σ can be obtained from that of the oxidized theory on $\hat{\Sigma}$ by dimensional reduction. In the latter theory, the only anomaly that can arise is due to the massless chiral fermions, and applying the standard index theorem as before we get

$$\text{index } i\hat{\mathcal{D}} = (-1)^{(p+2)(p+3)/2} \int_{\hat{\Sigma}} \text{ch}^+(\hat{\mathcal{E}}) \wedge \frac{\text{Td}(T\hat{\Sigma} \otimes \mathbf{C})}{\chi(T\hat{\Sigma})}. \tag{4.4}$$

Here the superbundle $\hat{E} = \hat{E}^+ \oplus \hat{E}^-$ is defined by $\hat{E}^\pm = \mathcal{S}^\pm(\hat{\Sigma}) \otimes \hat{E}_\rho$, and the Dirac operator $\hat{\mathcal{D}}$ defines the two-term complex

$$C^\infty(\hat{\Sigma}, \hat{E}^\pm) \xrightarrow{\hat{\mathcal{D}}} C^\infty(\hat{\Sigma}, \hat{E}^\mp). \tag{4.5}$$

The characteristic classes in (4.4) may be simplified as described in Section 3.1 to yield

$$\text{index } i\hat{\mathcal{D}} = \int_{\hat{\Sigma}} \text{ch}_\rho^+(\hat{E}) \wedge e^{d(N\hat{\Sigma})} \wedge \frac{\hat{A}(T\hat{\Sigma})}{\hat{A}(N\hat{\Sigma})} \wedge \chi(N\hat{\Sigma}). \tag{4.6}$$

The bundle \hat{E} is trivially \mathbf{Z}_2 -graded, and its Chern character may be represented by a closed differential form on $\hat{\Sigma}$,

$$\text{ch}_\rho^+(\hat{E}) = \text{ch}_\rho(\hat{E}) = \text{Tr}_\rho \exp \frac{1}{2\pi i} \hat{F}_{\hat{A}}, \tag{4.7}$$

where

$$\hat{F}_{\hat{A}} = d\hat{A} + [\hat{A} \wedge \hat{A}] \tag{4.8}$$

is the field strength of the gauge field (4.2).

The circle \mathbf{S}^1 is parallelizable in X , i.e. both its tangent and normal bundles are trivial so that $\chi(N\mathbf{S}^1) = \hat{A}(T\mathbf{S}^1) = \hat{A}(N\mathbf{S}^1) = 1$ and $d(N\mathbf{S}^1) = 0$. Using the multiplicativity (resp. additivity) of the characteristic classes $\hat{A}(V)$, $\chi(V)$ (resp. $d(V)$), and the decompositions $T\hat{\Sigma} = T\Sigma \oplus T\mathbf{S}^1$ and $N\hat{\Sigma} = N\Sigma \oplus N\mathbf{S}^1$, we find $\hat{A}(T\hat{\Sigma}) = \hat{A}(T\Sigma)$, and so on. It is now straightforward to dimensionally reduce the index integral (4.6) to the D-brane worldvolume Σ . We expand the fields on $\hat{\Sigma}$ in Fourier series around the \mathbf{S}^1 ,

$$\hat{A}_M(x, y) = \sum_{n=-\infty}^{\infty} A_M^{(n)}(x) e^{2\pi i n y}, \quad \hat{\Psi}(x, y) = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_1^{(n)}(x) \\ \psi_2^{(n)}(x) \end{pmatrix} e^{2\pi i n y}, \tag{4.9}$$

which upon integrating over the \mathbf{S}^1 part of the integral (4.6) will localize the fields onto their zero modes $A_\mu^{(0)}(x) = A_\mu(x)$, $A_y^{(0)}(x) = T(x)$ and $\psi_a^{(0)}(x) = \psi_a(x)$, $a = 1, 2$, on Σ . In particular, the curvature two-form (4.8) upon dimensionally reducing the fields becomes

$$\hat{F}_{\hat{A}} = F_A + D_A T \wedge dy, \tag{4.10}$$

where F_A is the field strength tensor of the original worldvolume gauge field A_μ and

$$D_A T = dT + [A, T] \tag{4.11}$$

is the gauge-covariant derivative of the tachyon field. Using (4.7) and (4.10), the \mathbf{S}^1 integration in (4.6) may be carried out explicitly to give

$$\begin{aligned} \oint_{\mathbf{S}^1} \text{ch}_\rho^+(\hat{E}) &= \sum_{k=1}^{\infty} \frac{1}{(2\pi i)^k} \frac{1}{k!} \oint_{\mathbf{S}^1} \text{Tr}_\rho (F_A + D_A T \wedge dy)^k \\ &= \sum_{k=1}^{\infty} \frac{1}{(2\pi i)^k} \frac{1}{(k-1)!} \text{Tr}_\rho ((F_A)^{k-1} \wedge D_A T) \\ &= \frac{1}{2\pi i} \text{Tr}_\rho \left(D_A T \wedge \exp \frac{1}{2\pi i} F_A \right). \end{aligned} \tag{4.12}$$

By using the Bianchi identity $D_A F_A = 0$, we arrive finally at

$$\text{index } i\hat{\mathcal{D}} = \frac{1}{2\pi i} \int_{\Sigma} d \text{Tr}_{\rho} \left(T \exp \frac{1}{2\pi i} F_A \right) \wedge e^{d(N\Sigma)} \wedge \frac{\hat{A}(T\Sigma)}{\hat{A}(N\Sigma)} \wedge \chi(N\Sigma). \quad (4.13)$$

As in Section 3.1, we may readily argue that the quantum anomaly arising from (4.13) can be cancelled by anomalous magnetic RR interactions. Comparing with (3.22) and incorporating the appropriate modifications described in Section 3.2, we arrive at the final form of the Chern–Simons action describing the anomalous coupling of a system of N non-BPS D-branes to the Ramond–Ramond fields C ,

$$\begin{aligned} \tilde{Z}_0 = & -\frac{1}{2\sqrt{2\pi}} \int_{\Sigma} \text{Tr} \phi^* \left(e^{i(\mathcal{J}_{\phi})^2 - [\mathcal{J}_{\phi}, T]} e^{\mathcal{J}_{\phi} d_{\perp}} \Big|_{x^{\perp}=0} \sum_{p=0}^{4+\sigma} C_{(2p+1-\sigma)} \wedge e^{-B/2\pi i} \right) \\ & \wedge D_A T \wedge \exp \frac{1}{2\pi i} F_A \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \wedge e^{d(N\Sigma)/2}, \end{aligned} \quad (4.14)$$

where here Tr denotes the (symmetrized) trace in the fundamental representation of the Chan–Paton gauge group $U(N)$. The $N \times N$ Hermitian matrices ϕ^i describe the transverse degrees of freedom of the non-BPS branes, and the commutator term $[\mathcal{J}_{\phi}, T]$ arises from the dimensional reduction of the pullback modification analogous to (3.24) and (3.25) involving the gauge-covariant derivative $\hat{D}_{\hat{A}}$. The subscript 0 in action (4.14) emphasizes the fact that it contains only the zero modes of the tachyon and gauge fields from the dimensional reduction, since in the calculation above we have simply eliminated the S^1 dependence of all fields before integrating over the extra dimension. A more precise evaluation should keep all higher Kaluza–Klein modes in (4.9) before integrating over the circle. However, in the present approach, it is difficult to keep track of these higher excitations, given that the index theory calculation relies on the structure of the lowest lying modes. The higher fermion modes, for example, have masses of order n^2 for the n th Kaluza–Klein state, and the appropriate anomaly cannot be identified for these massive fields, in the present energy regime that the calculations are based on. Furthermore, there is no immediate interpretation of these higher states in the original worldvolume field theory. To correctly account for the rest of the Kaluza–Klein spectrum, we will present another calculation of the anomalous coupling of non-BPS D-branes, which utilizes the previous superconnection formalism. This lends a more precise interpretation of the tachyon field which is based on the previous constructions.

4.2. Reduction from brane–antibrane pairs

In this section, we will begin by working in Type IIA superstring theory. An unstable IIA(B) Dp -brane may be realized as the projection of a IIB(A) Dp – $D\bar{p}$ system by the discrete \mathbf{Z}_2 symmetry generated by the operator $(-1)^{F_L}$ [28,33], where F_L is the left-moving part of the space–time fermion number operator. The operator $(-1)^{F_L}$ acts as multiplication by -1 on all Ramond sector states in the left-moving part of the fundamental string worldsheet, leaving all other sectors unchanged. It exchanges a D-brane with its antibrane so that

a brane–antibrane pair is invariant under $(-1)^{F_L}$ and it makes sense to take the \mathbf{Z}_2 quotient of this configuration. The feature that the $(-1)^{F_L}$ projection maps Type IIB superstring theory into Type IIA superstring theory can be proven using boundary states [59]. The fact that the brane–antibrane pair is mapped to a non-BPS D-brane follows from the action of the operator $(-1)^{F_L}$ on the Chan–Paton factors $\psi_{CP} \in U(2N)$ of the open strings in the Dp – $D\bar{p}$ system [28], which is given by

$$(-1)^{F_L} : \psi_{CP} \mapsto \sigma_1 \psi_{CP} \sigma_1, \tag{4.15}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \tag{4.16}$$

generates the one-dimensional complex Clifford algebra $C\ell_1^* = \mathbf{C} \oplus \mathbf{C}\sigma_1$.

As we discussed earlier, the lowest lying (GSO projected) bosonic states of a system of N brane–antibrane pairs wrapping a common worldvolume Σ may be geometrically encoded in the superconnection

$$\hat{\mathbb{A}} = d + \hat{\mathcal{A}} = \begin{pmatrix} d + \hat{A}^+ & \hat{T}^\dagger \\ \hat{T} & d + \hat{A}^- \end{pmatrix}, \tag{4.17}$$

where \hat{A}^\pm are the $U(N)$ gauge fields on the branes and antibranes, respectively, and \hat{T} is the bi-fundamental tachyon field from the open string p – \bar{p} states. The curvature of the superconnection (4.17) is

$$\hat{F}_{\hat{\mathbb{A}}} = \begin{pmatrix} \hat{F}^+ + \hat{T}^\dagger \hat{T} & \hat{D} \hat{T}^\dagger \\ \hat{D} \hat{T} & \hat{F}^- + \hat{T} \hat{T}^\dagger \end{pmatrix}. \tag{4.18}$$

Upon introducing the fields

$$\begin{aligned} A &= \frac{1}{2}(\hat{A}^+ + \hat{A}^-), & \bar{A} &= \frac{1}{2}(\hat{A}^+ - \hat{A}^-), \\ T &= \frac{1}{2}(\hat{T} + \hat{T}^\dagger), & \bar{T} &= \frac{1}{2}(\hat{T} - \hat{T}^\dagger), \end{aligned} \tag{4.19}$$

it follows that the only ones which survive the $(-1)^{F_L}$ projection are A and T , i.e. the quotient of the spectrum of the p – \bar{p} pairs sets $\bar{A} = \bar{T} = 0$. Clearly, the fermionic spectrum of the quotiented theory contains two Majorana–Weyl spinors of the same chirality, as the projection identifies the field contents on the branes and antibranes. In this way, we recover the low-energy spectrum of fields on the worldvolume Σ of a system of N non-BPS Dp -branes.

We may therefore compute the anomalous coupling of the unstable D-branes by taking the quotient of the anomaly term (3.14) for the brane–antibrane system. Evidently, the only change is the reduction of the Chern character, which with $\bar{A} = \bar{T} = 0$ in (4.18) becomes

$$[\text{ch}^+(\hat{E})]_{(-1)^{F_L}} = \text{Tr}(-1)^F \exp \frac{1}{2\pi i} \begin{pmatrix} F_A + T^2 & D_A T \\ D_A T & F_A + T^2 \end{pmatrix}, \tag{4.20}$$

where $\hat{E} = \hat{E}^+ \oplus \hat{E}^-$ is the Chan–Paton superbundle over the brane–antibrane worldvolume Σ , which reduces to a trivially \mathbf{Z}_2 -graded $U(N)$ gauge bundle $E \rightarrow \Sigma$ with corresponding worldvolume fields in (4.20) upon modding out by the operator $(-1)^{F_L}$ using its action (4.15) on the Chan–Paton factors. This \mathbf{Z}_2 -action induces an isomorphism $\hat{E}^+ \cong \hat{E}^- \equiv E$ and thereby identifies E as the diagonal sub-bundle of $[\hat{E}]_{(-1)^{F_L}} \cong E \oplus E$. Using the supersymmetric structure of (2.19) and (4.20) with $N^+ = N^- = N$, we may diagonalize the real, symmetric reduced superconnection field strength $[\hat{F}_{\hat{A}}]_{(-1)^{F_L}}$ to obtain

$$[\text{ch}^+(\hat{E})]_{(-1)^{F_L}} = \text{Tr} \left[\exp \frac{1}{2\pi i} (F_A + T^2 + D_A T) - \exp \frac{1}{2\pi i} (F_A + T^2 - D_A T) \right]. \tag{4.21}$$

Note that an elegant reduction such as (4.21) is not possible for the Chern character of a brane–antibrane system itself, whereby the superconnection curvature is generically a complex, Hermitian matrix with respect to the \mathbf{Z}_2 -grading.

The reduced Chern character (4.21) can be expanded into a more explicit expression by using the Dynkin form of the Baker–Campbell–Hausdorff formula [60]. This enables us to write

$$\begin{aligned} & \exp \frac{1}{2\pi i} (F_A + T^2 + D_A T) \\ &= \exp \left[\frac{1}{2\pi i} (T^2 + D_A T) + \sum_{r,s=1}^{\infty} \mathcal{E}_{rs}[A, T] \right] \wedge \exp \frac{1}{2\pi i} F_A, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \mathcal{E}_{rs}[A, T] &= \frac{(-1)^s}{(r+s)(2\pi i)^{r+s}} \sum_{m \geq 1} \frac{(-1)^m}{m} \\ &\times \sum_{\substack{p_1+q_1 \geq 1, \dots, p_m+q_m \geq 1 \\ p_1+\dots+p_m=r, q_1+\dots+q_m=s}} \frac{1}{p_1!q_1! \cdots p_m!q_m!} [(F_A + T^2 + D_A T)^{p_1} \wedge (F_A)^{q_1} \\ &\times \wedge [\cdots [(F_A + T^2 + D_A T)^{p_m} \wedge (F_A)^{q_m}] \cdots]]. \end{aligned} \tag{4.23}$$

The leading terms in the expansion of the right-hand side of (4.22) are given by

$$\begin{aligned} \mathcal{E}_2[A, T] &= -\frac{1}{2(2\pi i)^2} [F_A \wedge F_A + T^2 + D_A T], \\ \mathcal{E}_3[A, T] &= \frac{1}{12(2\pi i)^3} ([F_A + T^2 + D_A T \wedge F_A] \wedge F_A \\ &\quad + [F_A + T^2 + D_A T \wedge F_A] \wedge F_A + T^2 + D_A T), \\ \mathcal{E}_4[A, T] &= -\frac{1}{48(2\pi i)^4} ([F_A \wedge [F_A + T^2 + D_A T \wedge [F_A + T^2 + D_A T \wedge F_A]] \\ &\quad + [F_A + T^2 + D_A T \wedge [F_A \wedge [F_A + T^2 + D_A T \wedge F_A]]]), \end{aligned} \tag{4.24}$$

where $\mathcal{E}_n[A, T] = \sum_{r+s=n} \mathcal{E}_{rs}[A, T]$. Substituting (4.22) into (4.21), we arrive, in the usual way, at the complete RR coupling

$$\begin{aligned} \tilde{Z} = & -\frac{\sqrt{2\pi}}{2} \int_{\Sigma} \text{Tr} \phi^* \left(e^{i(\mathcal{J}_\phi)^2 - [\mathcal{J}_\phi, T]} e^{\mathcal{J}_\phi d_\perp} \Big|_{x^\perp=0} \sum_{p=0}^{4+\sigma} C_{(2p+1-\sigma)} \wedge e^{-B/2\pi i} \right) \\ & \wedge \left\{ \exp \left(\frac{1}{2\pi i} (T^2 + D_A T) + \sum_{r,s=1}^{\infty} \mathcal{E}_{rs}[A, T] \right) \right. \\ & \left. - \exp \left(\frac{1}{2\pi i} (T^2 - D_A T) + \sum_{r,s=1}^{\infty} \mathcal{E}_{rs}[A, -T] \right) \right\} \\ & \wedge \exp \frac{1}{2\pi i} F_A \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \wedge e^{d(N\Sigma)/2}, \end{aligned} \tag{4.25}$$

where $\phi^i = \frac{1}{2}(\hat{\phi}_+^i + \hat{\phi}_-^i)$ and now the transverse tachyon coupling $[\mathcal{J}_\phi, T]$ comes from the $(-1)^{F_L}$ projection of the operator (3.28). Action (4.25) is an odd function of the tachyon field T . Specifically, it contains only even powers of T and odd powers of $D_A T$. The expansion of (4.25) is similar in form to that constructed in [31], except that it generically contains extra powers of the field strength F_A coupled to the tachyon terms. To linear order in the tachyon field, action (4.25) coincides with the zero mode action (4.14) and hence the non-BPS D-brane coupling proposed in [30]. The complete series (4.25) thereby represents the contributions from all Kaluza–Klein sectors of the oxidized theory described in the previous section.

4.3. Tachyon condensation

Using action (3.33), it is possible to write down a natural geometric action for the system of unstable D-branes using the $(-1)^{F_L}$ projection. It is given by

$$\begin{aligned} \tilde{Z}_{\text{kin}} = & \frac{1}{2} \int_{\Sigma} [\|\hat{F}_{\hat{A}} - \hat{\mathcal{G}}\|^2]_{(-1)^{F_L}} \\ = & \int_{\Sigma} d \text{vol}(\Sigma) \text{Tr}[(F_A)^2 + (D_A T)^2 + (T^2 - \mathcal{G})^2 + \dots], \end{aligned} \tag{4.26}$$

where $\mathcal{G} = \frac{1}{2}(\hat{\mathcal{G}}^+ + \hat{\mathcal{G}}^-)$. The resulting tachyon potential in (4.26) has the anticipated \mathbf{Z}_2 reflection symmetry under the transformation $T \mapsto -T$ [10]. Again action (4.26) is minimized by flat gauge connections and covariantly constant tachyon fields. If $\mathcal{G} = m^2 \mathbb{1}$ with $m^2 > 0$, then the tachyon condensates obey the equation $T_c^2 = m^2 \mathbb{1}$ and the $U(N)$ gauge symmetry of the system is broken down to the subgroup $U(n_c) \times U(N - n_c)$, where n_c is the number of negative eigenvalues of T_c . Again this action is determined by a spectral Lagrangian of the form $\text{Tr}[\sigma \circ c^{-1}(\hat{\mathbb{V}}^2)]_{(-1)^{F_L}}^2$ involving the pertinent generalized Dirac operator. Thus for non-BPS D-branes, the same geometrical ingredients naturally lead to the standard processes involving tachyon condensation.

In the absence of a tachyon field the RR couplings (4.25) vanish, as expected since then the unstable D-brane configuration simply decays into the supersymmetric vacuum state. However, a topologically non-trivial tachyonic configuration can produce a lower-dimensional D-brane within the configuration of non-BPS branes. To demonstrate this, the crucial observation is that for processes involving tachyon condensation it is sufficient to focus on the zero mode part (4.14) of the total Chern–Simons action [31]. For a Higgs profile of the tachyon field, the terms in (4.25) involving higher powers of T or $D_A T$ will vanish. By dropping the non-abelian couplings to the transverse scalar fields, action (4.25) coincides with that of [30]. It is now straightforward to repeat the brane construction of Section 3.3 in the present case and induce the Chern–Simons term for a BPS D-brane wrapping a world-volume $\tilde{\Sigma}$ of co-dimension $2k + 1$ in Σ . For a Higgs-like configuration, action (4.25) can be reduced to the form

$$\begin{aligned} \tilde{Z}_0 = & -\frac{1}{2\sqrt{2\pi}} \int_{\tilde{\Sigma}} \sum_{p=0}^{4+\sigma} \phi^*(C_{(2p+1-\sigma)} \wedge e^{-B/2\pi i}) \wedge \text{ch}(\mathcal{L}) \wedge \sqrt{\frac{\hat{A}(T\tilde{\Sigma})}{\hat{A}(N\tilde{\Sigma})}} \\ & \wedge e^{d(N\tilde{\Sigma})/2} \int_{\tilde{\Sigma}'} d \text{Tr} \left(T \exp \frac{1}{2\pi i} F_A \right) \wedge \hat{A}[N(\tilde{\Sigma}, \Sigma)] \wedge e^{-c_1(L_N)/2}, \end{aligned} \quad (4.27)$$

where we have used the Bianchi identity. Here A is a connection on the twisted real spinor bundle $\mathcal{S}_{L_N^{1/2}}$ over $\tilde{\Sigma}$, while for the tachyon field configuration we have again taken Clifford multiplication

$$T(x) = \mathbb{1}_{\mathcal{L}} \otimes \left(f(x) \sum_{i=1}^{2k+1} \Gamma_i x^i \right) \oplus \mathbb{1}_I \quad (4.28)$$

with Γ_i the generators of the Clifford algebra of the transverse structure group $\text{SO}(2k + 1)$. From (4.27), it is evident that an appropriate choice of gauge connection A leads to the correct Ramond–Ramond coupling of a supersymmetric $D(p - 2k - 1)$ -brane, similarly to [30]. Since the characteristic classes are closed forms, the second integral in (4.27) can be reduced to the form $\oint_{\partial\tilde{\Sigma}'} \text{Tr}(T e^{F_A/2\pi i}) \wedge \hat{A} \wedge e^{-c_1/2}$. Since the tachyon field T is constant on $\partial\tilde{\Sigma}'$, we can choose A so that its field strength F_A has the appropriate generalized vortex configuration to yield a non-vanishing boundary integration.

5. Other non-supersymmetric brane systems

In this section, we will briefly explain how to obtain the anomalous couplings to all non-BPS systems of branes in Type II superstring theory. We shall do so by giving a set of rules for the transformations of the RR potentials in our previously derived Chern–Simons actions.

5.1. NS-branes

An important ingredient missing from our analysis of the Type IIB theory is its S-duality symmetry. This duality is also manifest at the level of p -brane solutions of 10-dimensional

supergravity. The Chern–Simons actions for unstable configurations of NS-branes in Type IIB superstring theory can be read off from the couplings (3.30) and (4.25) by applying the $SL(2, \mathbf{Z})$ S-duality transformation rules [61] to the total RR potential $C = \sum_p \phi^* C_{(p)}$, the B -field, and the field strengths F^\pm of the open fundamental strings ending on the D-branes. In the string frame, these transformations are given by

$$\begin{aligned} C_{(0)} &\mapsto -\frac{C_{(0)}}{(C_{(0)})^2 + e^{-2\varphi}}, & C_{(2)} &\mapsto B, & C_{(4)} &\mapsto C_{(4)}, & C_{(6)} &\mapsto B_{(6)}, \\ C_{(8)} &\mapsto -\tilde{C}_{(8)}, & B &\mapsto -C_{(2)}, & F^\pm &\mapsto \tilde{F}^\pm. \end{aligned} \quad (5.1)$$

Here φ is the dilaton field, $C_{(0)}$ the axion field, $B_{(6)}$ the electromagnetic dual of the NS–NS two-form B and \tilde{F}^\pm are the fluxes of the D1-branes that end on the NS-branes, which combine geometrically with the induced open p - \bar{p} D-string tachyon field \tilde{T} into the appropriate superconnection on the NS-brane worldvolume $\tilde{\Sigma}$. Note that the four-form RR potential is unaffected because of the self-duality of the D3-brane, while the transformation of the eight-form potential reflects the fact that D7-branes and NS7-branes do not form a doublet under S-duality [62]. The eight-form $\tilde{C}_{(8)}$ is related to the NS–NS and RR eight-forms by

$$d\tilde{C}_{(8)} = -C_{(0)} dB_{(8)} + [(C_{(0)})^2 + e^{-2\varphi}] dC_{(8)}. \quad (5.2)$$

The fields $(C_{(8)}, \tilde{C}_{(8)}, B_{(8)})$ thereby form a triplet under $SL(2, \mathbf{Z})$ transformations, where in addition to the transformation rule in (5.1) we have

$$\tilde{C}_{(8)} \mapsto -C_{(8)}, \quad B_{(8)} \mapsto -B_{(8)}. \quad (5.3)$$

Via T-duality, we also recover in this way the corresponding Chern–Simons actions for Type IIA NS-branes. The bound-state constructions of BPS NS-branes from unstable ones now also follow as outlined in Sections 3.3 and 4.3. Some details can be found in [56,63,64].

5.2. M -branes

The constructions of previous sections yield the anomalous couplings of all branes in Type II superstring theory, with the exception of the gravitational wave and the Kaluza–Klein monopole which are only defined in space–times that contain a special isometric direction. In the case of the pp-wave, this isometry lies in the direction of propagation of the wave, while for the KK-monopole it corresponds to the Taub–NUT fibre of its normal bundle. By oxidizing Type IIA superstring theory to M-Theory, these solitonic branes can be most naturally seen to arise from reductions of the corresponding M-branes in 11 dimensions. Nine-branes and ten-branes in M-Theory should, however, be dealt with in the context of massive 11-dimensional supergravity [65], since the BPS M9-brane couples magnetically to the mass field. While a fully covariant massive supergravity theory cannot be constructed in 11 dimensions [66], a supergravity action that is gauged with respect to an isometric 11th direction of space–time can be written down which reduces dimensionally to massive Type IIA Romans supergravity [67]. Reduction of the single M9-brane in this way along its gauged direction yields a D8-brane domain wall, while reduction along the transverse direction gives an NS9-brane. Reduction in another direction produces a KK8-brane monopole [62] with

a gauged direction in its worldvolume that is inherited from the M9-brane. In turn, these latter branes are most naturally understood as the electromagnetic duals of the mass field in massive Type IIA supergravity. Similarly, reduction of an M0-brane along the direction of its Killing vector yields a D0-brane, while its reduction in a non-isometric direction produces a gravitational wave.

The rules for uplifting the unstable Type IIA Chern–Simons actions (3.30) and (4.25) along the direction of a Killing vector field \hat{k} to M-Theory actions describing the couplings of unstable M-branes are given by [56,68]

$$\begin{aligned}
 C_{(1)} &\mapsto \frac{\hat{k}_{(1)}}{\|\hat{k}\|^2}, & C_{(3)} &\mapsto (\hat{C}_{(3)})_{\hat{\mu}\hat{\nu}\hat{\lambda}} \nabla^{\hat{k}} \hat{\mathcal{X}}^{\hat{\mu}} \wedge \nabla^{\hat{k}} \hat{\mathcal{X}}^{\hat{\nu}} \wedge \nabla^{\hat{k}} \hat{\mathcal{X}}^{\hat{\lambda}}, \\
 C_{(5)} &\mapsto \mathcal{J}_{\hat{k}} \hat{C}_{(6)} + \hat{C}_{(3)} \wedge \mathcal{J}_{\hat{k}} \hat{C}_{(3)}, & C_{(7)} &\mapsto \mathcal{J}_{\hat{k}} \hat{N}_{(8)}, & C_{(9)} &\mapsto \mathcal{J}_{\hat{k}} \hat{B}_{(10)}, \\
 B &\mapsto \mathcal{J}_{\hat{k}} \hat{C}_{(3)}, & F^{\pm} &\mapsto \hat{F}^{\pm}.
 \end{aligned}
 \tag{5.4}$$

The p -form fields on the right-hand side of (5.4) are assumed to be invariant under the isometry of space–time, $\mathcal{L}_{\hat{k}} \hat{C} = 0$ where $\mathcal{L}_{\hat{k}}$ is the Lie derivative along the Killing vector field \hat{k} , and the hats refer to 11-dimensional quantities. The one-form $\hat{k}_{(1)}$ is the Poincaré–Hodge dual to the Killing vector field, while $\hat{C}_{(3)}$ (resp. $\hat{C}_{(6)}$) is the usual three-form (resp. six-form) field of 11-dimensional supergravity. The matrix fields $\hat{\mathcal{X}}^{\hat{\mu}}$ are the non-abelian M-brane embedding coordinates in 11 dimensions, while $\nabla^{\hat{k}}$ is the usual non-abelian M-brane covariant derivative defined by

$$\nabla^{\hat{k}} \hat{\mathcal{X}}^{\hat{\mu}} = \hat{D} \hat{\mathcal{X}}^{\hat{\mu}} - \frac{\hat{k}_{\hat{\nu}} \hat{D} \hat{\mathcal{X}}^{\hat{\nu}}}{\|\hat{k}\|^2} \hat{k}^{\hat{\mu}},
 \tag{5.5}$$

where \hat{D} is the 11-dimensional gauge-covariant derivative defined analogously to (2.16). The eight-form field $\hat{N}_{(8)}$ is the Hodge dual of the Killing one-form $\hat{k}_{(1)}$, while $\hat{B}_{(10)}$ is the electromagnetic dual of the mass field. The gauge field curvatures \hat{F}^{\pm} are the fluxes of the M2-branes wrapped around the direction of \hat{k} , which induce a $p-\bar{p}$ tachyon field \hat{T} in the M-brane worldvolume $\hat{\Sigma}$ that combine into the appropriate superconnection. The KK-monopole and pp-wave couplings can now be obtained by oxidizing the appropriate Type IIA actions using the transformation rules (5.4), and then dimensionally reducing them in a worldvolume direction. The corresponding transformations of all objects appearing in the Chern–Simons actions (3.30) and (4.25) can be worked out in the same way as the other dimensional reductions discussed in this paper. Further details can be found in [56,63,69], where the corresponding bound-state constructions are also given. In fact, all Type II branes can be obtained via the totality of reductions and tachyon condensations on the oxidized M-brane couplings described in this section [56,63].

6. K-theory analysis

Let Σ be a compact spin^c -brane–antibrane worldvolume manifold in Type II superstring theory, and let $E = E^+ \oplus E^-$ be the corresponding Chan–Paton superbundle over

Σ . Under the usual physical assumptions of brane–antibrane creation and annihilation [1,2,9,10,54], the details of which have been substantially verified via boundary string field theory calculations [4], the Chan–Paton bundles on the branes and antibranes should be subjected to the equivalence relation $(E^+, E^-) \sim (E^+ \oplus H, E^- \oplus H)$ for all gauge bundles H , and hence the net D-brane charge of this configuration depends only on its K-theory class $[(E^+, E^-)] = [E^+] - [E^-] \in K^0(\Sigma)$. By using the Thom isomorphism and the Atiyah–Bott–Shapiro construction [9,10], it is possible to map this class to an element of the K-theory group $K^0(X)$ of space–time. This utilizes the construction that was presented in Section 3.3 which derived D-brane charge from a higher-dimensional brane–antibrane system. In this section, we will show how this fact ties in very naturally with the superconnection formalism developed in this paper in terms of Dirac operators and index theory. This identification is supported by the analysis of the previous section, which shows that *all* branes are universally classified by the appropriate K-theory groups [64,69].

An interpretation of the anomalous coupling in (3.1) and (3.2) for supersymmetric systems of D-branes, which also takes into account the subtleties associated with the self-duality of the RR fields [24], has been given in terms of K-theory in [25]. One can define a K-theory group $\hat{K}^{p+1}(X)$ of p -form fields, analogously to the framework of Deligne cohomology, via the exact sequence

$$0 \rightarrow \frac{K^p(X, \mathbf{R})}{K^p(X)} \rightarrow \hat{K}^{p+1}(X) \rightarrow B^{p+1}(X) \rightarrow 0, \tag{6.1}$$

where $K^p(X, \mathbf{R}) = K^p(X) \otimes \mathbf{R}$ and

$$B^{p+1}(X) = \{(x, \omega) \in K^{p+1}(X) \times H^{p+1}(X, \mathbf{Z}) \mid \text{ch}(x) = [\omega]_{\text{DR}}\} \tag{6.2}$$

with $[\omega]_{\text{DR}} \in H^{p+1}(X, \mathbf{R})$ the de Rham representative of the integer cohomology class ω . For a system of BPS D p -branes, we then have $e^{iZ} \in \hat{K}^{p+1}(X)$, where the type of K-theory (complex, real or quaternionic) depends on the value of $p \bmod 8$ [25]. However, in what follows we shall see that the analysis of the present paper is more naturally connected to K-homology, showing that K-homology is really the appropriate setting for the topological classification of D-brane charge. This has been pointed out previously in different contexts, and from very different points of view, in [10,36,37]. We shall only deal with the construction of K-theory classes over the worldvolume Σ , as then the mapping to the K-theory of space–time can be carried through by using standard techniques.

6.1. Index bundles and Chern–Simons couplings

In this section, we shall work in Type IIB superstring theory so that $\dim \Sigma = p + 1$ is even. Consider the Clifford bundle $C\ell(\Sigma)$ over Σ , and let \not{D} be the corresponding Dirac operator. By using Swan’s theorem, we can represent the Chan–Paton superbundle over Σ as the range $E = \Pi \mathcal{O}_N(\Sigma)$ of a projection $\Pi : \mathcal{O}_N(\Sigma) \rightarrow \mathcal{O}_N(\Sigma)$, $\Pi^2 = \Pi = \Pi^\dagger$, of the trivial vector bundle $\mathcal{O}_N(\Sigma) \rightarrow \Sigma$ of rank $N = (p + 1)\text{ch}_0(E)$. The corresponding twisted Dirac operator \not{D}_E on $C\ell(\Sigma)$ with coefficients in E may thereby be expressed as

$$\not{D}_E = \Pi(\not{D} \otimes \mathbb{1})\Pi. \tag{6.3}$$

With $\mathcal{H} = C^\infty(\Sigma, S_E) = \mathcal{H}^+ \oplus \mathcal{H}^-$ the graded Hilbert space of smooth E -valued spinor fields on Σ , the Dirac operator is a map $\mathcal{D}_E : \mathcal{H}^\pm \rightarrow \mathcal{H}^\mp$ and so it can be decomposed with respect to the \mathbf{Z}_2 -grading as

$$\mathcal{D}_E = \begin{pmatrix} 0 & \mathcal{D}_E^- \\ \mathcal{D}_E^+ & 0 \end{pmatrix}. \tag{6.4}$$

If ∇ is an ordinary connection on E , then the quantity

$$\mathbb{E} = (-1)^F \otimes \nabla + \mathcal{D}_E = \begin{pmatrix} \nabla & \mathcal{D}_E^- \\ \mathcal{D}_E^+ & -\nabla \end{pmatrix} \tag{6.5}$$

defines a superconnection of the twisted spinor bundle $S_E = S_E^+ \oplus S_E^-$. The Chern character of S_E is then given by [11]

$$\text{ch}^+(S_E) = \text{ch}^+(\mathcal{H}, \mathcal{D}_E) = \text{Tr} \exp \frac{1}{2\pi i} F_{\mathbb{E}}, \tag{6.6}$$

where in the first equality we have emphasized the fact that ch^+ depends only on the choice of Dirac operator \mathcal{D}_E acting on a particular graded Hilbert space \mathcal{H} . Formula (6.6) follows from the one-to-one correspondence between generalized Dirac operators and superconnections that we mentioned in Section 2.3, and the Lichnerowicz formula (3.37). Moreover, the cohomology class of (6.6) does not depend on the off-diagonal odd parts of the superconnection (6.5), and so it determines an element $\text{ch}(E^+) - \text{ch}(E^-) \in H^{\text{even}}(\Sigma, \mathbf{Q})$. In other words, the Chern character depends only on the choice of virtual bundle $[(E^+, E^-)] \in \mathbf{K}^0(\Sigma)$. Going back to the superconnection \mathbb{A} introduced in Section 2.1, this simply means that the induced D-brane charge is independent of the choice of profile for the tachyon field T . This is precisely what was found in Section 3.3 via explicit calculation.

The key feature here is that the anomalous coupling on the brane–antibrane worldvolume is determined entirely by the choice of Dirac operator, or more precisely by the pair $(\mathcal{H}, \mathcal{D}_E)$. As we will now discuss, this immediately leads to the relationship to K-theory, or more precisely to K-homology. The family of finite-dimensional subspaces $\ker i\mathcal{D}_E^\pm \subset \mathcal{H}^\mp$ defines a virtual bundle over Σ known as the index bundle [10,70,71]

$$\text{Ind}(\mathcal{H}, \mathcal{D}_E) = [\ker i\mathcal{D}_E^+] - [\ker i\mathcal{D}_E^-] \in \mathbf{K}^0(\Sigma). \tag{6.7}$$

The closed differential form (6.6) is then also a representative of the Chern character $\text{ch}^+(\text{Ind}(\mathcal{H}, \mathcal{D}_E)) \in H^{\text{even}}(\Sigma, \mathbf{Q})$ of the index bundle. By using this property and the generic, untwisted Dirac operator \mathcal{D} , we may define a natural pairing on K-theory known as the index map

$$\text{Index}_{\mathcal{D}} : \mathbf{K}^0(\Sigma) \rightarrow \mathbf{Z}, \tag{6.8}$$

which is given by

$$\text{Index}_{\mathcal{D}}([E]) = \text{index } i\mathcal{D}_E = \text{ch}_0(\text{Ind}(\mathcal{H}, \mathcal{D}_E)) = \dim \ker i\mathcal{D}_E^+ - \dim \ker i\mathcal{D}_E^-. \tag{6.9}$$

Therefore, the anomaly arising from the Dirac spinor fields on Σ leads naturally to a pairing on K-theory. This pairing in turn defines K-homology, as we will discuss in the next section.

Before doing this, let us first show precisely how the index bundle is related to the generalized Chern–Simons forms which are used to generate the Ramond–Ramond couplings, and hence the appropriate pairing on K-theory. We write the generalized Dirac operator $\mathcal{D}_E = Q(\nabla^s \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A}) \equiv \mathcal{V}^s + \mathcal{A}$ as in Section 2.3, and introduce a linear homotopy of superconnections. For fixed $t \in [0, 1]$, we consider the superconnection

$$\mathbb{E}(t) = \delta_{\text{BRST}} + \mathcal{V}^s + t(\Lambda + \mathcal{A}), \tag{6.10}$$

where Λ is the Cartan–Maurer form (2.47). Its curvature is

$$\mathbb{E}(t)^2 = (t^2 - t)(\Lambda^2 + [\Lambda, \mathcal{A}]) + (\mathcal{V}^s + t\mathcal{A})^2, \tag{6.11}$$

which, at $t = 0, 1$, obeys the horizontality condition [11]

$$\mathbb{E}(0)^2 = Q \circ \omega(R_\nabla), \quad \mathbb{E}(1)^2 = F_{\mathbb{E}}. \tag{6.12}$$

The superconnection (6.10) thereby defines a continuous interpolation between the index bundle and the Clifford bundle over Σ .

The Chern character (6.6) may then be used to construct generalized Chern–Simons forms via the generating function

$$\xi(\mathcal{H}, \mathcal{D}) = \int_0^1 dt \text{Tr}^+(\Lambda + \mathcal{A}) \exp \frac{1}{2\pi i} \mathbb{E}(t)^2. \tag{6.13}$$

The relevant part of (6.13) insofar as the chiral gauge anomaly is concerned is the degree 1 component in the BRST ghost field Λ . To find it, we use the Duhamel expansion

$$e^{A+B} = e^A + \sum_{n \geq 1} \int_{\Delta_n} dt_0 e^{t_0 A} \prod_{a=1}^n dt_a B e^{t_a A}, \tag{6.14}$$

where

$$\Delta_n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbf{R}^{n+1} \mid t_a \geq 0, \sum_a t_a = 1 \right\} \tag{6.15}$$

is the standard n -simplex in \mathbf{R}^{n+1} . We set $A = (1/2\pi i)(\mathcal{V}^s + t\mathcal{A})^2$ and $B = ((t^2 - t)/2\pi i)(\omega^2 + [\omega, \mathcal{A}])$ in (6.14). By using (6.11), we then find that the term in (6.13) which is linear in the BRST ghost field comes solely from the leading and $n = 1$ terms in expansion (6.14). In this way, we arrive at the total Chern–Simons form

$$\begin{aligned} \xi^{(1)}(\mathcal{H}, \mathcal{D}) = \int_0^1 dt \text{Tr}^+ \left[\Lambda \exp \frac{1}{2\pi i} (\mathcal{V}^s + t\mathcal{A})^2 + \frac{t^2 - t}{2\pi i} \mathcal{A} \int_0^1 dt' \right. \\ \left. \left(\exp \frac{t'}{2\pi i} (\mathcal{V}^s + t\mathcal{A})^2 \right) [\Lambda, \mathcal{A}] \left(\exp \frac{t' - 1}{2\pi i} (\mathcal{V}^s + t\mathcal{A})^2 \right) \right]. \end{aligned} \tag{6.16}$$

Note that when $\mathcal{A} = R_\nabla = 0$, (6.16) coincides with the Cartan form $\text{Tr}^+ \Lambda$. In the general case, the form $\xi^{(1)}(\mathcal{H}, \mathcal{D})$ is a deformation of the Cartan co-cycle which has the same group cohomology class. While this provides a nice characterization of the topological anomaly,

the important aspect is that the inhomogeneous form (6.13), which essentially determines the form \mathcal{Y} to which the Ramond–Ramond potentials couple, involves a continuous deformation to the index bundle generated by the Dirac operator \mathcal{D} , and is thereby naturally related to K-homology, as we now explain.

6.2. Fredholm modules and bivariant K-theory

To place the discussion of the previous section into a precise K-theoretical framework, we shall use the dual, algebraic characterization of the geometry of the brane–antibrane worldvolume Σ in terms of the algebra $\mathbf{A} = C^\infty(\Sigma)$ of smooth complex-valued functions on Σ . We represent \mathbf{A} on the Hilbert space \mathcal{H} diagonally by pointwise multiplication of functions. The K-theory group $K^0(\Sigma)$ may be defined as the space of equivalence classes of projections Π acting on \mathcal{H} . Two projections Π and Π' are said to be (algebraically) equivalent if there is a partial isometry U on \mathcal{H} with $\Pi = U^\dagger U$ and $\Pi' = UU^\dagger$. The K-homology group $K_0(\Sigma)$ is now defined in terms of Fredholm operators \mathcal{F} acting on the Hilbert space \mathcal{H} , i.e. those operators for which there exists another operator \mathcal{Q} such that $\mathcal{F}\mathcal{Q} - \mathbb{1}$, $\mathcal{Q}\mathcal{F} - \mathbb{1}$ and $[\mathcal{F}, f]^+ \forall f \in \mathbf{A}$ are all elements of the elementary algebra $\mathcal{K}(\mathcal{H})$ of compact operators on \mathcal{H} . We further assume that the operator \mathcal{F} is odd with respect to the \mathbf{Z}_2 -grading on \mathcal{H} , i.e. $\varepsilon\mathcal{F} = -\mathcal{F}\varepsilon$. The pair $(\mathcal{H}, \mathcal{F})$ is called an even K-cycle and the quadruple $(\mathbf{A}, \mathcal{H}, \mathcal{F}, \varepsilon)$ is known as an even Fredholm module [71]. The abelian group $K_0(\Sigma)$ may be represented in terms of homotopy classes of K-cycles with respect to direct sum. The natural pairing between K-theory and K-homology is then provided by the index map which generalizes (6.3), (6.8) and (6.9),

$$\begin{aligned} K^0(\Sigma) \times K_0(\Sigma) &\rightarrow \mathbf{Z}, \\ ([\Pi], [\mathcal{F}]) &\mapsto \text{index } \Pi\mathcal{F}\Pi. \end{aligned} \tag{6.17}$$

Note that any pair $(\mathcal{H}, \mathcal{D})$ determines a Fredholm module [72]. While the Dirac operator \mathcal{D} is unbounded, the commutators $[\mathcal{D}, f]^+$ are bounded and $f(\mathbb{1} + \mathcal{D}^- \mathcal{D}^+)^{-1} \in \mathcal{K}(\mathcal{H})$ for all $f \in \mathbf{A}$. Then

$$\mathcal{F} = \frac{\mathcal{D}^+}{\sqrt{\mathbb{1} + \mathcal{D}^- \mathcal{D}^+}} = \mathcal{D}^+ \int_0^\infty \frac{ds}{\sqrt{s}} \frac{1}{s + \mathbb{1} + \mathcal{D}^- \mathcal{D}^+} \tag{6.18}$$

is a Fredholm operator. Conversely, any Fredholm module can be obtained in this way (up to homotopy). The pair $(\mathcal{H}, \mathcal{D})$ is therefore usually referred to as a Dirac K-cycle and it is the underlying analytical object which generates K-homology.

The natural pairing between K-theory and K-homology is actually best understood through a bivariant form of K-theory known as KK-theory [73]. The concept of Fredholm module naturally extends to that of a Kasparov module which is a quintuple $(\mathbf{A}, \mathbf{B}, \mathcal{H}, \mathcal{F}, \varepsilon)$ where \mathbf{A} and \mathbf{B} are algebras. The generalization which occurs is that while \mathbf{A} is still represented on \mathcal{H} by bounded operators, \mathcal{H} is now a (right) Hilbert module over \mathbf{B} , i.e. a right \mathbf{B} -module which admits an inner product with values in \mathbf{B} and which is complete with respect to this inner product. The remaining properties are as in the case of Fredholm modules. The abelian group of homotopy classes of Kasparov modules with respect to direct sum defines the KK-group $KK_0(\mathbf{A}, \mathbf{B})$. The functor $\mathbf{A} \mapsto KK_0(\mathbf{A}, \mathbf{B})$ is covariant while

$B \mapsto KK_0(A, B)$ is contravariant. In particular, when $B = C$ the group $KK_0(A, C)$ is by definition just the abelian group of homotopy classes of Fredholm modules over the algebra A . Specializing to the case $A = C^\infty(\Sigma)$, we thereby have

$$KK_0(C^\infty(\Sigma), C) = K_0(\Sigma). \tag{6.19}$$

On the other hand, with $A = C$ the group $KK_0(C, B)$ is the abelian group of equivalence classes of (right) projective B -modules, and again specializing to $B = C^\infty(\Sigma)$ we have by the Serre–Swan theorem that

$$KK_0(C, C^\infty(\Sigma)) = K^0(\Sigma). \tag{6.20}$$

Therefore, we see that the Dirac K -cycle $(\mathcal{H}, \mathcal{D})$ interpolates between the K -homology $[(\mathcal{H}, \mathcal{D})] \in KK_0(C^\infty(\Sigma), C)$ and K -theory $\text{Ind}(\mathcal{H}, \mathcal{D}_E) \in KK_0(C, C^\infty(\Sigma))$ groups.

The real advantage of the KK -theory description is that there is typically a line bundle Π which is a projection on $M_N(C^\infty(\Sigma) \otimes \bar{A})$, with M_N the algebra of $N \times N$ matrices with entries in the given algebra such that the index bundle of K -theory can be represented as [71,73]

$$\text{Ind}(\mathcal{H}, \mathcal{D}) = [\Pi] \otimes_{\bar{A}} (\mathcal{H}, \mathcal{D}) \in KK_0(C, C^\infty(\Sigma)) \tag{6.21}$$

in terms of the Kasparov product $\otimes_{\bar{A}}$ of Π by the K -homology cycle $(\mathcal{H}, \mathcal{D})$. The Kasparov product here is a map

$$\otimes_{\bar{A}} : KK_0(C, C^\infty(\Sigma) \otimes \bar{A}) \times KK_0(\bar{A}, C) \rightarrow KK_0(C, C^\infty(\Sigma)), \tag{6.22}$$

where \bar{A} is the L^∞ -norm closure of the algebra $A = C^\infty(\Sigma)$ in the C^* -algebra of bounded linear operators on the separable Hilbert space \mathcal{H} . The result (6.21) emphasizes the fact that K -homology, through the Dirac K -cycle $(\mathcal{H}, \mathcal{D})$, is the defining topological property of D-brane charge.

The Kasparov product is most elegantly described by introducing the notion of a Cuntz algebra as follows [74]. Since Σ is assumed to be compact, $A = C^\infty(\Sigma)$ is a unital algebra. The Cuntz algebra QA is defined to be the free product $QA = A \otimes A$ in the category of unital algebras, i.e. with amalgamation over the identity 1_A of A . Then QA is naturally a super-algebra, and there is a canonical “folding” homomorphism $\varphi : QA \rightarrow A$ which identifies the two copies of the algebra A inside QA . Let $qA = \ker \varphi$. Given a Kasparov module $(A, B, \mathcal{H}, \mathcal{F}, \varepsilon)$ we can induce a homomorphism $\alpha : qA \rightarrow \mathcal{K}(\mathcal{H})$ by

$$\alpha(\eta) = P_{\text{GSO}}\eta, \quad \alpha^\vee(\eta) = P_{\text{GSO}}\mathcal{F}\eta\mathcal{F}, \tag{6.23}$$

where, for any $f \in A$, $f \mapsto f \otimes 1_A$ and $f^\vee \mapsto 1_A \otimes f$ are the two canonical monomorphisms $A \hookrightarrow QA$. To characterize the relationship between Cuntz algebras and KK -theory, we denote by $[qA, \mathcal{K}(\mathcal{H}) \otimes B]$ the semi-group of homotopy classes of algebra homomorphisms $\alpha : qA \rightarrow \mathcal{K}(\mathcal{H}) \otimes B$ with respect to the direct sum $\alpha \oplus \alpha' : qA \rightarrow M_2(\mathcal{K}(\mathcal{H}) \otimes B) \cong \mathcal{K}(\mathcal{H}) \otimes B$, where the isomorphism is a consequence of Morita equivalence. Then, for any two algebras A and B , one can show that [74]

$$KK_0(A, B) = [qA, \mathcal{K}(\mathcal{H}) \otimes B]. \tag{6.24}$$

The result (6.24) may now be used to define the Kasparov product $KK_0(A, B) \times KK_0(B, C) \rightarrow KK_0(A, C)$ for any three algebras A, B and C .

If ζ is the canonical generator of $K_0(q\mathbf{C}) \cong K_0(\mathbf{C}) \cong \mathbf{Z}$, then to any element $[\alpha] \in [q\mathbf{C}, \mathcal{K}(\mathcal{H}) \otimes A]$ we can assign the K-theory class

$$K_0(\alpha)[\zeta] \in K_0(\mathcal{K}(\mathcal{H}) \circ \otimes A) \cong K_0(A), \tag{6.25}$$

where we have used the stability of K-theory under Morita equivalence. With $A = \mathbf{C}$ and $B = C^\infty(\Sigma)$ we then arrive at the isomorphism

$$[q\mathbf{C}, \mathcal{K}(\mathcal{H}) \otimes C^\infty(\Sigma)] \cong K^0(\Sigma), \tag{6.26}$$

showing how the Cuntz algebra description (6.24) naturally achieves the desired K-theory and K-homology interpolation. The natural \mathbf{Z}_2 -grading on the Cuntz algebra QA or on qA fits in nicely with the fact that Σ is the worldvolume of a brane–antibrane pair, and this natural association gives rise to the K-homology group $K_0(\Sigma)$. The crucial feature here though is that the operator (6.5) is a Quillen superconnection and so the corresponding index theorem characterizes the cohomology class of the topological anomaly, represented most naturally through the Dirac K-cycle (\mathcal{H}, \not{D}) .

The Ramond–Ramond fields $G = dC$ also define elements of K-theory through the modified Bianchi identity $dG = 2\pi \operatorname{ch}^+(x) \wedge \sqrt{\hat{A}(TX)}$ which leads to the field

$$G(x) = 2\pi \operatorname{ch}^+(x) \wedge \sqrt{\hat{A}(TX)} \tag{6.27}$$

that depends only on the virtual bundle $x = [(E^+, E^-)]$ [24,25]. The anomalous couplings $\int_\Sigma C \wedge \mathcal{Y}$ represent a pairing between K-theory and K-homology, or equivalently a natural pairing on bivariant KK-theory. Alternatively, we may view it as a pairing between de Rham cohomology and K-homology through the homological Chern character. By using the push-forward ϕ_* induced on homology by the embedding $\Sigma \hookrightarrow \phi X$, we may lift all of these statements to find that D-brane charge in Type IIB superstring theory is labelled by the K-homology group $K_0(X) = \operatorname{KK}_0(C^\infty(X), \mathbf{C})$ of the space–time manifold X . However, this interpretation does not explain why the geometrical Dirac genus term appears with a square root in the pairing, as in (3.22) or (6.27), and the appearance of this factor is one of the fundamental aspects of the K-theoretic (or otherwise) formulation of D-brane charges and Ramond–Ramond fields for which a heuristic interpretation is still lacking.

Analogous conclusions for Type IIA D-branes, for which $\dim \Sigma = p + 1$ is odd, can be reached by removing the grading ε into positive and negative chirality spinor fields from the definitions of Fredholm and Kasparov modules above. The corresponding equivalence classes of odd modules define the higher K-homology and KK-groups $K_1(\Sigma)$ and $\operatorname{KK}_1(A, B)$, respectively. Again by using push-forward maps ϕ_* we may infer that D-brane charge in Type IIA superstring theory takes values in the K-homology group $K_1(X) = \operatorname{KK}_1(C^\infty(X), \mathbf{C})$ of space–time. Note that the distinction between Type IIA and Type IIB K-groups is far more natural in K-homology than it is in the K-theory of virtual bundles, because it relies solely on the dimensionality of the D-brane worldvolume to determine a chirality grading on the corresponding space of spinor fields. It would be interesting to find an interpretation of the intermediary bivariant K-theory groups $\operatorname{KK}_0(A, B)$ which interpolate between the K-homology and K-theory groups.

6.3. Toeplitz operators and unstable D-branes

Unstable Dp -branes in Type II superstring theory have been interpreted as stringy analogues of sphalerons [75], i.e. static classical solutions with a single negative fluctuation mode. They are thereby related to the non-trivial homotopy of string configuration space, and also intimately to K-theory. A BPS $D(p - 1)$ -brane, whose charge is classified by K-theory, gives rise to a one-parameter family of static configurations whose topology is the same as that of the stable $D(p - 1)$ -brane. The extra parameter lives in a manifold \mathcal{M} which replaces the Euclidean time and parametrizes the D-sphalerons. The unstable Dp -branes thereby produce a family of virtual bundles $[(E^+(t), E^-(t))] = x(t) \in K^0(X), t \in \mathcal{M}$ so that the homotopy of string configuration space on the space–time manifold X is related to the K-theory group $K^0(X \times \mathcal{M})$. The connection to K-theory using superconnections and index theory, or equivalently Dirac operators, in this case is most naturally done using the dimensional reduction method of Section 4.1. This interpretation is very much in the same spirit as the sphaleron picture of unstable branes and it yields an analytical description of the RR couplings that were derived in Section 4.

We work now in Type IIA superstring theory and consider the oxidation (4.1) of the even-dimensional worldvolume Σ of the unstable D-branes. With the Hilbert space $\mathcal{H}_{S^1} = L^2(S^1, dy)$, the Dirac K-cycle of the circle S^1 parametrized by $y \in [0, 1]$ is $(\mathcal{H}_{S^1}, d/dy)$. Its Chern character in homology is the fundamental class $[S^1] \in H_1(S^1, \mathbb{Z})$. We are thereby led to consider the K-homology cycle of $\hat{\Sigma}$ which is the graded tensor product $(\mathcal{H}', \mathcal{D}') = (\mathcal{H}, \mathcal{D}) \otimes (\mathcal{H}_{S^1}, d/dy)$, where $\mathcal{H}' = \mathcal{H} \otimes \mathcal{H}_{S^1}$ and the Dirac operator of this product is given by

$$\mathcal{D}' = (-1)^F \otimes \frac{d}{dy} + \mathcal{D} \otimes 1 = \begin{pmatrix} \frac{d}{dy} & \mathcal{D}^- \\ \mathcal{D}^+ & -\frac{d}{dy} \end{pmatrix}. \tag{6.28}$$

The pair $(\mathcal{H}', \mathcal{D}')$ thus defines an odd K-cycle, i.e. it determines an element of $K_1(\hat{\Sigma})$, and decomposing this group using the Künneth theorem and Poincaré duality gives

$$K_1(\Sigma \times S^1) = K_1(\Sigma) \oplus K_0(\Sigma), \tag{6.29}$$

thereby determining an element of $K_1(\Sigma)$ via projection onto the first summand of (6.29). The details of this projection are equivalent to the elimination of all higher winding modes of the fields that was done in the derivation of Section 4.1. This follows from the fact that the canonical projection map $\pi : \hat{\Sigma} \rightarrow \Sigma$ induces an epimorphism $\pi_* : K_1(\hat{\Sigma}) \rightarrow K_1(\Sigma)$ with $\ker \pi_* = K_0(\Sigma)$ so that the group $K_0(\Sigma)$ accounts for the winding modes around the circle S^1 of $\hat{\Sigma}$.

We can now understand the role of these winding modes more precisely as follows. The components of the Chern character of $(\mathcal{H}', \mathcal{D}')$ are given by [76]

$$\text{ch}_n^+(\mathcal{H}', \mathcal{D}') [f_0, f_1, \dots, f_n] = \lambda_n \int_{\hat{\Sigma}} \hat{A}(T\hat{\Sigma}) \wedge \text{Tr}(f_0 df_1 \wedge \dots \wedge df_n), \tag{6.30}$$

where λ_n are some universal coefficients and $f_a \in \mathbb{M}_N(C^\infty(\Sigma \times S^1))$. The topological anomaly ξ_A corresponding to the Cartan–Maurer form $\Lambda = g^{-1} \delta_{\text{BRST}} g$ will then be

determined, as described in Sections 2.4 and 6.1, by a sum over the Chern characters (6.30) of odd degree $n = 2m + 1$ obtained by setting $f_0 = g^{-1}$, $f_1 = g$, \dots , $f_{2m} = g^{-1}$, $f_{2m+1} = g$. Since $\hat{A}(T\hat{\Sigma}) = \hat{A}(T\Sigma)$, this leads to the following expression [51]:

$$\xi_\Lambda = \sum_{m \geq 0} (-1)^m m! \lambda_{2m+1} \int_\Sigma \hat{A}(T\Sigma) \oint_{\mathbf{S}^1} \text{Tr}(g^{-1} dg \wedge dg^{-1} \wedge \dots \wedge dg). \tag{6.31}$$

The integral over the extra dimension \mathbf{S}^1 in (6.31) is most elegantly understood through the formalism of Toeplitz operators, as we now describe.

Let us consider first the case of a single unstable D-brane, i.e. $N = 1$. The Hilbert space $\mathcal{H}_{\mathbf{S}^1}$ is a module over the algebra $\mathbf{A}_{\mathbf{S}^1} = C^\infty(\mathbf{S}^1)$, with the action of $\mathbf{A}_{\mathbf{S}^1}$ on $\mathcal{H}_{\mathbf{S}^1}$ given by pointwise multiplication of functions. The Hardy space $\mathcal{H}_{\mathbf{S}^1}^+$ is defined to be the L^2 -norm closure in $\mathcal{H}_{\mathbf{S}^1}$ of the linear span of the set of functions $\{e^{2\pi i n y}\}_{n \geq 0}$, i.e.

$$\mathcal{H}_{\mathbf{S}^1}^+ = \overline{\bigoplus_{n=0}^\infty \mathbf{C} e^{2\pi i n y}}. \tag{6.32}$$

Let $P_+ : \mathcal{H}_{\mathbf{S}^1} \rightarrow \mathcal{H}_{\mathbf{S}^1}^+$ be the corresponding orthogonal projection. For any function $f \in \mathbf{A}_{\mathbf{S}^1}$, we may associate a Toeplitz operator on $\mathcal{H}_{\mathbf{S}^1}$ by

$$\mathcal{T}_f = P_+ f P_+, \tag{6.33}$$

which, with respect to the orthogonal decomposition $\mathcal{H}_{\mathbf{S}^1} = \mathcal{H}_{\mathbf{S}^1}^+ \oplus (\mathbb{1} - P_+) \mathcal{H}_{\mathbf{S}^1}$, can be expressed as

$$\mathcal{T}_f = \begin{pmatrix} P_+ f P_+ & P_+ f (\mathbb{1} - P_+) \\ (\mathbb{1} - P_+) f P_+ & (\mathbb{1} - P_+) f (\mathbb{1} - P_+) \end{pmatrix}. \tag{6.34}$$

For any $f \in \mathbf{A}_{\mathbf{S}^1}$, \mathcal{T}_f is a bounded operator on $\mathcal{H}_{\mathbf{S}^1}$. If f is further an invertible function on \mathbf{S}^1 , then the Toeplitz operator $\mathcal{T}_f : \mathcal{H}_{\mathbf{S}^1}^+ \rightarrow \mathcal{H}_{\mathbf{S}^1}^+$ is a Fredholm operator whose index is given by

$$\text{index } \mathcal{T}_f = \text{Tr}_{P_+ \mathcal{H}_{\mathbf{S}^1}} (\mathbb{1} - \mathcal{T}_{f^{-1}} \mathcal{T}_f) - \text{Tr}_{P_+ \mathcal{H}_{\mathbf{S}^1}} (\mathbb{1} - \mathcal{T}_f \mathcal{T}_{f^{-1}}). \tag{6.35}$$

It is then a straightforward calculation that establishes the index theorem [77]

$$\text{index } \mathcal{T}_f = \frac{1}{2\pi i} \oint_{\mathbf{S}^1} f^{-1} df, \tag{6.36}$$

which relates the index of the Toeplitz operator \mathcal{T}_f to the winding number of the function $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$.

By considering the Hilbert space $\mathcal{H}_{\mathbf{S}^1}^+ \otimes \mathbf{C}^N$, this construction may be easily generalised to matrix-valued functions $f : \mathbf{S}^1 \rightarrow \mathbb{M}_N(\mathbf{C})$, and hence to a multi-brane system [37]. In particular, we may apply an index theorem of the type (6.36) to the winding numbers of the tachyon field on $\hat{\Sigma}$, regarded as a function $y \mapsto \hat{A}_y(x, y)$ on $\mathbf{S}^1 \rightarrow U(N)$. By comparing this result with (6.31), we may now put the index theoretical calculation of Section 4.1 into a proper K-theoretic interpretation. Namely, the dimensional reduction (obtained by eliminating the \mathbf{S}^1 -dependence of all fields on $\hat{\Sigma}$) corresponds to the incorporation of Toeplitz operators of index zero in the Chern character mapping of the

K-homology cycle over $\hat{\Sigma}$ which defines the appropriate index class. However, the crucial property again is that the operator (6.28) is in fact a Quillen superconnection that acts on smooth sections of the family of Hilbert spaces \mathcal{H} over S^1 . Therefore, the pertinent index theorem that was used in Section 4.1 to yield the unstable D-brane charge actually computes the overall number of eigenvalues of \mathcal{D}' which cross zero in a homotopy between elements in its gauge orbit. The index theorem thereby again characterizes the cohomology class of the topological anomaly, or equivalently of the D-brane charge.

It is instructive to see how this structure arises within the formalism of the brane–antibrane reduction of Section 4.2. The Chan–Paton gauge bundle $E \rightarrow \Sigma$ on the unstable Type IIA D-branes is ungraded. However, the Clifford algebra $Cl_1^* = \mathbf{C} \oplus \mathbf{C}\sigma_1$ has a natural \mathbf{Z}_2 -grading, and so the product $\hat{E} = E \otimes Cl_1^* = \hat{E}^+ \oplus \hat{E}^-$ is a superbundle which is identified with the Chan–Paton vector bundle of the corresponding Type IIB brane–antibrane pairs. Correspondingly, the endomorphism algebra of \hat{E} is the superbundle

$$\text{End } \hat{E} \cong \text{End}(E) \otimes Cl_1^*, \tag{6.37}$$

which gives rise to superconnections $\hat{A} \in \Omega^-(\Sigma, \text{End } \hat{E})$ of the form $\hat{A} = \mathbb{1}_{2N} \otimes D_A + \sigma_1 \otimes T$. The associated generalized Dirac operators are

$$\hat{\mathcal{D}} = \begin{pmatrix} \mathcal{D}_A & \mathcal{G}_s \otimes T \\ \mathcal{G}_s \otimes T & \mathcal{D}_A \end{pmatrix}. \tag{6.38}$$

The grading automorphism of the superbundle \hat{E} is the generator (4.16) of the Clifford algebra Cl_1^* , and it commutes with the Dirac operator (6.38). Thus the pair $(\mathcal{H}, \hat{\mathcal{D}})$ defines an *odd* K-homology cycle, and hence an element of $K_1(\Sigma)$. As with the derivations of Section 4, the relationship between the unstable brane charges and the higher K-homology group of the worldvolume Σ comes about in a much more direct way through the reduction from brane–antibrane pairs. The dimensional reduction formalism does, however, expose the physical meaning of the oxidation.

Acknowledgements

The author thanks E. Akhmedov, F. Lizzi, J. Mickelsson, R. Nest, N. Obers, K. Olsen, B. Schroers, A. Schwarz and S. Willerton for helpful discussions, and F. Larsen for very helpful comments on the manuscript. A preliminary version of this paper was presented at the APCTP-KIAS Winter School and Workshop on “Strings and D-Branes 2000” in Seoul, Korea, February 17–25, 2000. Part of this work was carried out during the PIMS/APCTP/PITP Frontiers of Mathematical Physics Workshop on “Particles, Fields and Strings” at Simon Fraser University, Vancouver, Canada, July 16–27, 2001. This work was supported in part by an Advanced Fellowship from the Particle Physics and Astronomy Research Council (UK).

References

- [1] A. Sen, Non-BPS states and branes in string theory, in: L. Baulieu, M. Green, M. Picco, P. Windey (Eds.), *Cargèse 1999: Progress in String Theory and M-Theory*, Kluwer Academic Publishers, Dordrecht, 2001, p. 187. hep-th/9904207;
A. Lerda, R. Russo, Stable non-BPS states in string theory: a pedagogical review, *Int. J. Mod. Phys. A* 15 (2000) 771. hep-th/9905006;
J.H. Schwarz, TASI lectures on non-BPS D-brane systems, in: L. Baulieu, M. Green, M. Picco, P. Windey (Eds.), *Cargèse 1999: Progress in String Theory and M-Theory*, Kluwer Academic Publishers, Dordrecht, 2001, p. 147. hep-th/9908144;
M.R. Gaberdiel, Lectures on non-BPS Dirichlet branes, *Class. Quant. Grav.* 17 (2000) 3483. hep-th/0005029.
- [2] A. Sen, Tachyon condensation on the brane–antibrane system, *J. High Energy Phys.* 9808 (1998) 12. hep-th/9805170.
- [3] A. Sen, B. Zwiebach, Tachyon condensation in string field theory, *J. High Energy Phys.* 3 (2000) 2. hep-th/9912249;
N. Berkovits, The tachyon potential in open Neveu–Schwarz string field theory, *J. High Energy Phys.* 4 (2000) 22. hep-th/0001084;
N. Moeller, W. Taylor, Level truncation and the tachyon in open bosonic string field theory, *Nucl. Phys. B* 583 (2000) 105. hep-th/0002237.
- [4] A.A. Gerasimov, S.L. Shatashvili, On exact tachyon potential in open string field theory, *J. High Energy Phys.* 10 (2000) 34. hep-th/0009103;
D. Kutasov, M. Mariño, G. Moore, Some exact results on tachyon condensation in string field theory, *J. High Energy Phys.* 10 (2000) 45. hep-th/0009148;
D. Kutasov, M. Mariño, G. Moore, Remarks on tachyon condensation in superstring field theory. hep-th/0010108;
D. Ghoshal, A. Sen, Normalization of the background-independent open string field theory action, *J. High Energy Phys.* 11 (2000) 21. hep-th/0009191;
V. Niarchos, N. Prezas, Boundary superstring field theory, *Nucl. Phys. B* 619 (2001) 51. hep-th/0103102.
- [5] P. Kraus, F. Larsen, Boundary string field theory of the $D\bar{D}$ system, *Phys. Rev. D* 63 (2001) 106004. hep-th/0012198.
- [6] T. Takayanagi, S. Terashima, T. Uesugi, Brane–antibrane action from boundary string field theory, *J. High Energy Phys.* 103 (2001) 19. hep-th/0012210.
- [7] O.D. Andreev, Some computations of partition functions and tachyon potentials in background-independent off-shell string theory, *Nucl. Phys. B* 598 (2001) 151. hep-th/0010218;
A.A. Tseytlin, Sigma model approach to string theory effective actions with tachyons, *J. Math. Phys.* 42 (2001) 2854. hep-th/0011033.
- [8] R. Minasian, G. Moore, K-theory and Ramond–Ramond charge, *J. High Energy Phys.* 9711 (1997) 2. hep-th/9710230.
- [9] E. Witten, D-branes and K-theory, *J. High Energy Phys.* 9812 (1998) 19. hep-th/9810188.
- [10] K. Olsen, R.J. Szabo, Constructing D-branes from K-theory, *Adv. Theoret. Math. Phys.* 3 (1999) 889. hep-th/9907140.
- [11] D. Quillen, Superconnections and the Chern character, *Topology* 24 (1985) 89.
- [12] V. Mathai, D. Quillen, Superconnections, Thom classes and equivariant differential forms, *Topology* 25 (1986) 85.
- [13] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer, Berlin, 1992.
- [14] C. Kennedy, A. Wilkins, Ramond–Ramond couplings on brane–antibrane systems, *Phys. Lett. B* 464 (1999) 206. hep-th/9905195.
- [15] M. Alishahiha, H. Ita, Y. Oz, On superconnections and the tachyon effective action, *Phys. Lett. B* 503 (2001) 181. hep-th/0012222.
- [16] J.H. Schwarz, E. Witten, Anomaly analysis of brane–antibrane systems, *J. High Energy Phys.* 103 (2001) 32. hep-th/0103099.
- [17] E.T. Akhmedov, A.A. Gerasimov, S.L. Shatashvili, On unification of RR couplings, *J. High Energy Phys.* 107 (2001) 40. hep-th/0105228.

- [18] M. Li, Boundary states of D-branes and Dy strings, *Nucl. Phys. B* 460 (1996) 351. hep-th/9510161;
M. Bershadsky, V. Sadov, C. Vafa, D-branes and topological field theories, *Nucl. Phys. B* 463 (1996) 420. hep-th/9511222;
M.R. Douglas, Branes within branes, in: L. Baulieu, P. Di Francesco, M.R. Douglas, V.A. Kazakov, M. Picco, P. Windey (Eds.), *Strings, Branes and Dualities*, Kluwer Academic Publishers, Dordrecht, 1999, p. 267. hep-th/9512077.
- [19] M.B. Green, J.A. Harvey, G. Moore, I-brane inflow and anomalous couplings on D-branes, *Class. Quant. Grav.* 14 (1997) 47. hep-th/9605033.
- [20] Y.-K.E. Cheung, Z. Yin, Anomalies, branes and currents, *Nucl. Phys. B* 517 (1998) 69. hep-th/9710206.
- [21] C.A. Scrucca, M. Serone, Anomalies and inflow on D-branes and O-planes, *Nucl. Phys. B* 556 (1999) 197. hep-th/9903145.
- [22] E. Witten, Baryons and branes in anti-de Sitter space, *J. High Energy Phys.* 9807 (1998) 6. hep-th/9805112.
- [23] D.S. Freed, E. Witten, Anomalies in string theory with D-branes. hep-th/9907189.
- [24] G. Moore, E. Witten, Self-duality, Ramond–Ramond fields and K-theory, *J. High Energy Phys.* 5 (2000) 32. hep-th/9912279.
- [25] D.S. Freed, M.J. Hopkins, On Ramond–Ramond fields and K-theory, *J. High Energy Phys.* 5 (2000) 44. hep-th/0002027.
- [26] W. Taylor, M. Van Raamsdonk, Multiple Dp -branes in weak background fields, *Nucl. Phys. B* 573 (2000) 703. hep-th/9910052;
R.C. Myers, Dielectric branes, *J. High Energy Phys.* 9912 (1999) 22. hep-th/9910053;
M.R. Garousi, R.C. Myers, Worldvolume potentials on D-branes, *J. High Energy Phys.* 11 (2000) 32. hep-th/0010122.
- [27] S.F. Hassan, R. Minasian, D-brane couplings, RR fields and Clifford multiplication. hep-th/0008149.
- [28] A. Sen, BPS D-branes on non-supersymmetric cycles, *J. High Energy Phys.* 9812 (1998) 21. hep-th/9812031.
- [29] P. Hořava, Type IIA D-branes, K-theory and matrix theory, *Adv. Theoret. Math. Phys.* 2 (1998) 1373. hep-th/9812135.
- [30] M. Billò, B. Craps, F. Roose, Ramond–Ramond couplings of non-BPS D-branes, *J. High Energy Phys.* 9906 (1999) 33. hep-th/9905157.
- [31] J. Klusoň, D-branes in type IIA and type IIB theories from tachyon condensation. hep-th/0001123.
- [32] B. Janssen, P. Meesen, A nonabelian Chern–Simons term for non-BPS D-branes, *J. High Energy Phys. Conf. Proc.* tnr2000/030. hep-th/0009025;
S. Mukhi, N.V. Suryanarayana, Chern–Simons terms on noncommutative branes, *J. High Energy Phys.* 11 (2000) 6. hep-th/0009101.
- [33] O. Bergman, K. Hori, P. Yi, Confinement on the brane, *Nucl. Phys. B* 580 (2000) 289. hep-th/0002223.
- [34] A. Sen, Supersymmetric worldvolume action for non-BPS D-branes, *J. High Energy Phys.* 9910 (1999) 8. hep-th/9909062.
- [35] N.D. Lambert, I. Sachs, Nonabelian field theory of stable non-BPS D-branes, *J. High Energy Phys.* 3 (2000) 28. hep-th/0002061;
M.R. Garousi, Tachyon couplings on non-BPS D-branes and Dirac–Born–Infeld action, *Nucl. Phys. B* 584 (2000) 284. hep-th/0003122;
E.A. Bergshoeff, M. de Roo, T.C. de Wit, E. Eyras, S. Panda, T-duality and actions for non-BPS D-branes, *J. High Energy Phys.* 5 (2000) 9. hep-th/0003221;
J. Klusoň, Proposal for non-BPS D-brane action, *Phys. Rev. D* 62 (2000) 126003. hep-th/0004106.
- [36] V. Periwal, D-brane charges and K-homology, *J. High Energy Phys.* 7 (2000) 41. hep-th/0006223;
V. Periwal, Deformation quantization as the origin of D-brane nonabelian degrees of freedom, *J. High Energy Phys.* 8 (2000) 21. hep-th/0008046;
Y. Matsuo, Topological charges of noncommutative soliton, *Phys. Lett. B* 499 (2001) 223. hep-th/0009002.
- [37] J.A. Harvey, G. Moore, Noncommutative tachyons and K-theory, *J. Math. Phys.* 42 (2001) 2765. hep-th/0009030.
- [38] Y. Ne’eman, S. Sternberg, Internal supersymmetry and superconnections, *Proc. Natl. Acad. Sci.* 87 (1990) 7875;
C.-Y. Lee, Y. Ne’eman, Superconnections and electroweak symmetry breaking, *Phys. Lett. B* 264 (1991) 389;
R. Coquereaux, G. Esposito-Farèse, G. Vaillant, Higgs fields as Yang–Mills fields and discrete symmetries, *Nucl. Phys. B* 353 (1991) 689;

- J. Tolksdorf, The Einstein–Hilbert–Yang–Mills–Higgs action and the Dirac–Yukawa operator, *J. Math. Phys.* 39 (1998) 2213. hep-th/9612149;
- H. Figueroa, J.M. Gracia-Bondía, F. Lizzi, J.C. Várilly, A nonperturbative form of the spectral action principle in noncommutative geometry, *J. Geom. Phys.* 26 (1998) 329. hep-th/9701179;
- G. Roepstorff, Superconnections and the Higgs field, *J. Math. Phys.* 40 (1999) 2698. hep-th/9801040.
- [39] T. Banks, L. Susskind, Brane–antibrane forces. hep-th/9511194.
- [40] M.B. Green, Pointlike states for type IIB superstrings, *Phys. Lett. B* 329 (1994) 435. hep-th/9403040;
- M.B. Green, M. Gutperle, Light cone supersymmetry and D-branes, *Nucl. Phys. B* 476 (1996) 484. hep-th/9604091;
- G. Lifschytz, Comparing D-branes to black-branes, *Phys. Lett. B* 388 (1996) 720. hep-th/9604156;
- V. Periwal, Antibranes and crossing symmetry. hep-th/9612215;
- M. Srednicki, IIB or not IIB, *J. High Energy Phys.* 9808 (1998) 5. hep-th/9807138.
- [41] C. Vafa, Brane–antibrane systems and $U(N|M)$ supergroup. hep-th/0101218.
- [42] S.S. Chern, J. Simons, Characteristic forms and geometric invariants, *Ann. Math.* 99 (1974) 48.
- [43] J. Wess, B. Zumino, Consequences of anomalous ward identities, *Phys. Lett. B* 37 (1971) 95;
- B. Zumino, Cohomology of gauge groups: co-cycles and Schwinger terms, *Nucl. Phys. B* 253 (1985) 477.
- [44] J. Cheeger, J. Simons, Differential characters and geometric invariants, in: *Geometry and Topology, Lecture Notes in Mathematics*, Vol. 1167, Springer, Berlin, 1985, p. 50.
- [45] P. Mora, H. Nishino, Fundamental extended objects for Chern–Simons supergravity, *Phys. Lett. B* 482 (2000) 222. hep-th/0002077;
- P. Mora, Chern–Simons supersymmetric branes, *Nucl. Phys. B* 594 (2001) 229. hep-th/0008180.
- [46] D.S. Freed, Determinants, torsion and strings, *Commun. Math. Phys.* 107 (1986) 483.
- [47] M.F. Atiyah, I.M. Singer, The index of elliptic operators 5, *Ann. Math.* 93 (1971) 139.
- [48] L. Alvarez-Gaumé, Supersymmetry and the Atiyah–Singer index theorem, *Commun. Math. Phys.* 90 (1983) 161;
- D. Friedan, P. Windey, Supersymmetric derivation of the Atiyah–Singer index and the chiral anomaly, *Nucl. Phys. B* 235 (1984) 395.
- [49] R.L. Bryant, E. Sharpe, D-branes and Spin^c structures, *Phys. Lett. B* 450 (1999) 353. hep-th/9812084.
- [50] M.F. Atiyah, I.M. Singer, The index of elliptic operators on compact manifolds, *Bull. Am. Math. Soc.* 69 (1963) 422.
- [51] M.F. Atiyah, I.M. Singer, Dirac operators coupled to vector bundles, *Proc. Natl. Acad. Sci.* 81 (1984) 2597.
- [52] L. Alvarez-Gaumé, P. Ginsparg, The structure of gauge and gravitational anomalies, *Ann. Phys.* 161 (1985) 423.
- [53] I. Pesando, On the effective potential of the $Dp\text{--}\overline{Dp}$ system in type II theories, *Mod. Phys. Lett. A* 14 (1999) 1545. hep-th/9902181.
- [54] A. Sen, $\text{SO}(32)$ spinors of type I and other solitons on brane–antibrane pair, *J. High Energy Phys.* 9809 (1998) 023. hep-th/9808141.
- [55] K. Hori, D-branes, T-duality and index theory, *Adv. Theoret. Math. Phys.* 3 (1999) 281. hep-th/9902102.
- [56] Y. Lozano, Noncommutative branes from M-theory, *Phys. Rev. D* 64 (2001) 106011. hep-th/0012137.
- [57] E. Witten, Some exact multi-instanton solutions of classical Yang–Mills theory, *Phys. Rev. Lett.* 38 (1977) 121;
- N.S. Manton, A new six-dimensional approach to the Weinberg–Salam model, *Nucl. Phys. B* 158 (1979) 141;
- D.B. Fairlie, Higgs fields and the determination of the Weinberg angle, *Phys. Lett. B* 82 (1979) 97.
- [58] A. Sen, Type I D-particle and its interactions, *J. High Energy Phys.* 9810 (1998) 21. hep-th/9809111.
- [59] K. Dasgupta, J. Park, Brane mapping under $(-1)^{F_L}$, *Phys. Lett. B* 457 (1999) 47. hep-th/9903125.
- [60] V.S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Springer, New York, 1984.
- [61] E.A. Bergshoeff, E. Eyras, R. Halbersma, J.P. van der Schaar, C.M. Hull, Y. Lozano, Space–time filling branes and strings with sixteen supercharges, *Nucl. Phys. B* 564 (2000) 29. hep-th/9812224.
- [62] E. Eyras, Y. Lozano, Exotic branes and nonperturbative seven-branes, *Nucl. Phys. B* 573 (2000) 735. hep-th/9908094.
- [63] L. Houart, Y. Lozano, Type II branes from brane–antibrane in M-theory, *Nucl. Phys. B* 575 (2000) 195. hep-th/9910266.
- [64] L. Houart, Y. Lozano, S-duality and brane descent relations, *J. High Energy Phys.* 3 (2000) 31. hep-th/9911173.
- [65] E.A. Bergshoeff, Y. Lozano, T. Ortin, Massive branes, *Nucl. Phys. B* 518 (1998) 363. hep-th/9712115.

- [66] K. Bautier, S. Deser, M. Henneaux, D. Seminara, No cosmological $D = 11$ supergravity, *Phys. Lett. B* 409 (1997) 49. hep-th/9704131.
- [67] L.J. Romans, Massive $N = 2A$ supergravity in ten dimensions, *Phys. Lett. B* 169 (1986) 374.
- [68] E.A. Bergshoeff, E. Eyras, Y. Lozano, The massive Kaluza–Klein monopole, *Phys. Lett. B* 430 (1998) 77. hep-th/9802199;
T. Sato, A ten-form gauge potential and an M9-brane Wess–Zumino action in massive 11-D theory, *Phys. Lett. B* 477 (2000) 457. hep-th/9912030.
- [69] L. Houart, Y. Lozano, Brane descent relations in M-theory, *Phys. Lett. B* 479 (2000) 299. hep-th/0001170.
- [70] M.F. Atiyah, *K-theory*, Benjamin, New York, 1967.
- [71] A. Connes, *Noncommutative Geometry*, Academic Press, New York, 1994.
- [72] S. Baaj, P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens, *Comput. Rend. Acad. Sci. Paris* 296 (1983) 875.
- [73] G. Skandalis, Kasparov’s bivariant K-theory and applications, *Exposit. Math.* 9 (1991) 193.
- [74] J. Cuntz, Quantized differential forms in noncommutative differential topology and geometry, *Contemp. Math.* 145 (1993) 65.
- [75] J.A. Harvey, P. Hořava, P. Kraus, D-sphalerons and the topology of string configuration space, *J. High Energy Phys.* 3 (2000) 21. hep-th/0001143;
N. Drukker, D.J. Gross, N. Itzhaki, Sphalerons, merons and unstable branes in AdS, *Phys. Rev. D* 62 (2000) 86007. hep-th/0004131;
S. Corley, S. Ramgoolam, Projector equivalences in K-theory and families of noncommutative solitons, *J. High Energy Phys.* 103 (2001) 37. hep-th/0012217.
- [76] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, *Geom. Funct. Anal.* 5 (1995) 174.
- [77] U. Venugopalkrishna, Fredholm operators associated with strongly pseudo-convex domains in \mathbf{C}^n , *J. Funct. Anal.* 9 (1972) 349.